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OCCURRENCE OF ELLIPTICAL FRACTAL PATTERNS IN MULTI-BIT BANDPASS SIGMA DELTA MODULATORS

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It has been established that the class of bandpass sigma delta modulators (SDMs) with single bit quantizers could exhibit state space dynamics represented by elliptic or fractal patterns confined within trapezoidal regions. In this letter, we find that elliptical fractal patterns may also occur in bandpass SDMs with multibit quantizers, even for the case when the saturation regions of the multibit quantizers are not activated and a large number of bits are used for the implementation of the quantizers. Moreover, the fractal pattern may occur for low bit quantizers, and the visual appearance of the phase portraits between the infinite state machine and the finite state machine with high bit quantizers is different. These phenomena are different from those previously reported for the digital filter with two's complement arithmetic. Furthermore, some interesting phenomena are found. A bit change of the quantizer can result in a dramatic change in the fractal patterns. When the trajectories of the corresponding linear systems converge to a fixed point, the regions of the elliptical fractal patterns diminish in size as the number of bits of the quantizers increases.

Keywords: Bandpass sigma delta modulator; multibit; fractal behaviors.

1. Introduction

Sigma Delta modulation achieves A/D and D/A conversion by using some very simple and low cost components [Steiner & Yang, 1997]. Hence, SDMs are common in many industrial and engineering applications. Research and development of SDMs are concerned with the use of multibit quantizers because SDMs with single bit quantizers are highly unstable and typically overloaded [Lipshitz & Vanderkooy, 2000].

It is well known that elliptical fractal patterns may occur in bandpass SDMs with single bit quantizers [Feely, 1995]. The question arises whether similar patterns will occur for the multibit case. If the saturation regions of the quantizers are not activated and there are an infinite number of bits for the implementation of quantizers, then the bandpass SDMs become linear systems and fractal behavior would not occur. Consequently, one may ask if the number of bits of quantizers is increased, but the saturation regions of the quantizers are still not activated, would the nonlinear behavior disappear? If not, what behaviors would be shown on the phase portraits as the number of bits of quantizers is increased?

The organization of the letter is as follows. The system is described in Sec. 2. Some results are shown in Sec. 3. The conclusion is summarized in Sec. 4.

2. System Description

Consider the bandpass SDM discussed in [Feely, 1995]. It can be described by the following state space equation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{s}(k) + \mathbf{C}\mathbf{u}(k) \quad \text{for } k \geq 0, \tag{1}$$

where $\mathbf{x}(k) \equiv [x_1(k) \quad x_2(k)]^T$ is the state vector function of the system, $\mathbf{u}(k) \equiv [u(k-2) \quad u(k-1)]^T$ is a vector containing the past two consecutive points from the input signal $u(k)$,

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 2\cos\theta \end{bmatrix}, \quad \text{and} \qquad \mathbf{B} \equiv \mathbf{C} \equiv \begin{bmatrix} 0 & 0 \\ -1 & 2\cos\theta \end{bmatrix} \tag{2}$$

in which the superscript ^{T} denotes the transpose operator and $\theta \in (-\pi, \pi)\backslash\{0\}$. When $\theta \in \{-\pi, 0, \pi\}$, the system is either a lowpass or highpass SDM, which is out of scope of this letter. We consider the case when $\mathbf{x}(k)$ and $u(k)$ are real signals and $u(k)$ is a constant input, that is $\mathbf{u}(k) = \mathbf{u}$ for $k \geq 0$.

For the N-bit bandpass SDMs,

$$\mathbf{s}(k) \equiv [Q(x_1(k)) \quad Q(x_2(k))]^T \quad \text{for } k \geq 0, \tag{3}$$

where Q is a uniform midrise quantizer and represented as

$$Q(y) \equiv \begin{cases} \dfrac{y}{|y|} & |y| > L \\ 0 & y = 0 \\ \dfrac{y\Delta}{|y|}\text{ceil}\left(\dfrac{|y|}{\Delta}\right) & |y| \leq L \text{ and } y \neq 0 \end{cases}, \tag{4}$$

in which Δ is the step size of the quantizers and defined as

$$\Delta \equiv \frac{1}{2^{N-1}}, \tag{5}$$

$$L \equiv \Delta(2^{N-1} - 1), \tag{6}$$

$|y|$ denotes the absolute value of y and ceil (y) is the nearest integer of y towards infinity. $\mathbf{s}(k)$ is a vector containing discrete output sequences. The values of $\mathbf{s}(k)$ can be viewed as symbols, so $\mathbf{s}(k)$ is called a symbolic sequence.

Once the initial conditions $\mathbf{x}(0)$, the filter parameters of the system θ, the input $\mathbf{u}$ and the number of bits of the quantizers N are given, the state vector function $\mathbf{x}(k)$ and the symbolic sequences $\mathbf{s}(k)$ (or the output sequences of the systems) can be uniquely determined by Eq. (1).

3. Results

Figures 1(a)–1(c) show the phase portraits of a bandpass SDM and Figs. 1(d)–1(f) show the shifted phase portraits of a bandpass SDM with

$$\theta = \cos^{-1}(-0.158532), \quad \mathbf{u} = -0.3[1 \quad 1]^T, \quad \mathbf{x}(0) = [0 \quad 0.5]^T \tag{7}$$

and different values of N. The values of the state variables are bounded by -1 and 1. This means the system is operating in the quantization region,

$$|x_i(k)| \leq L \quad \text{for } k \geq 0 \text{ and for } i = 1, 2, \tag{8}$$

and the saturation regions ($|x_i(k)| > L$) are not activated. Therefore, as the number of bits of the quantizers is increased, the bandpass SDM is a closer approximation of the corresponding linear system. So by estimating the behavior of the system using the corresponding linear model, the system is expected to converge to a fixed point located at $[0.4 \quad 0.4]^T$. However, elliptical fractal patterns are exhibited on the phase portrait even when a large number of bits ($N = 37$) are used for the implementation of the quantizers.

According to this result, the visual appearance of the phase portraits between the infinite state machine and the finite state machine with high bit quantizers is different. This result is different from the existing results on second-order digital filters with two's complement arithmetic [Lin & Chua, 1991], where there are visually indistinguishable elliptic fractal patterns shown on the phase portraits when 16 bits are used for the implementation of the quantizers. Besides, the fractal pattern may

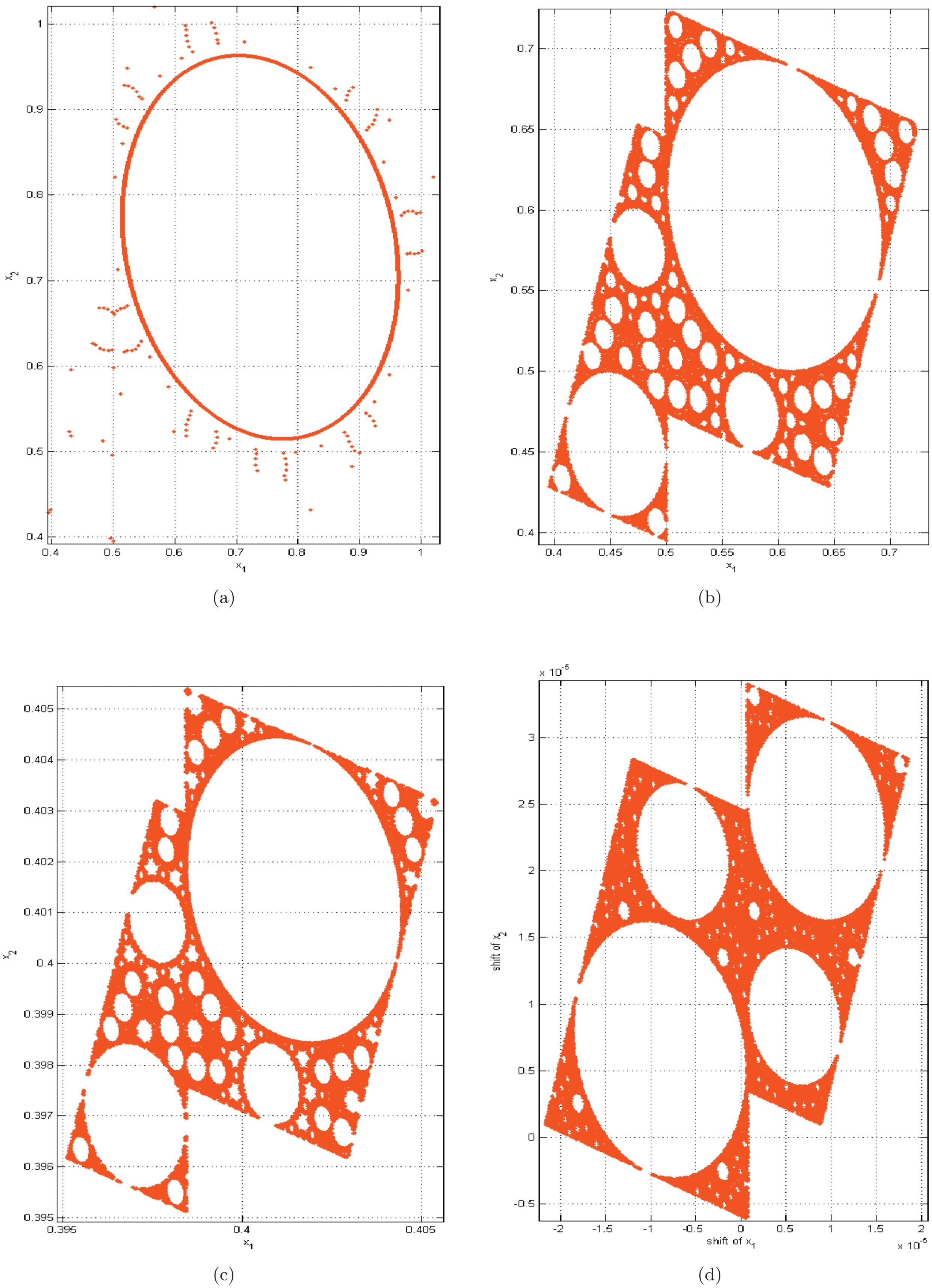
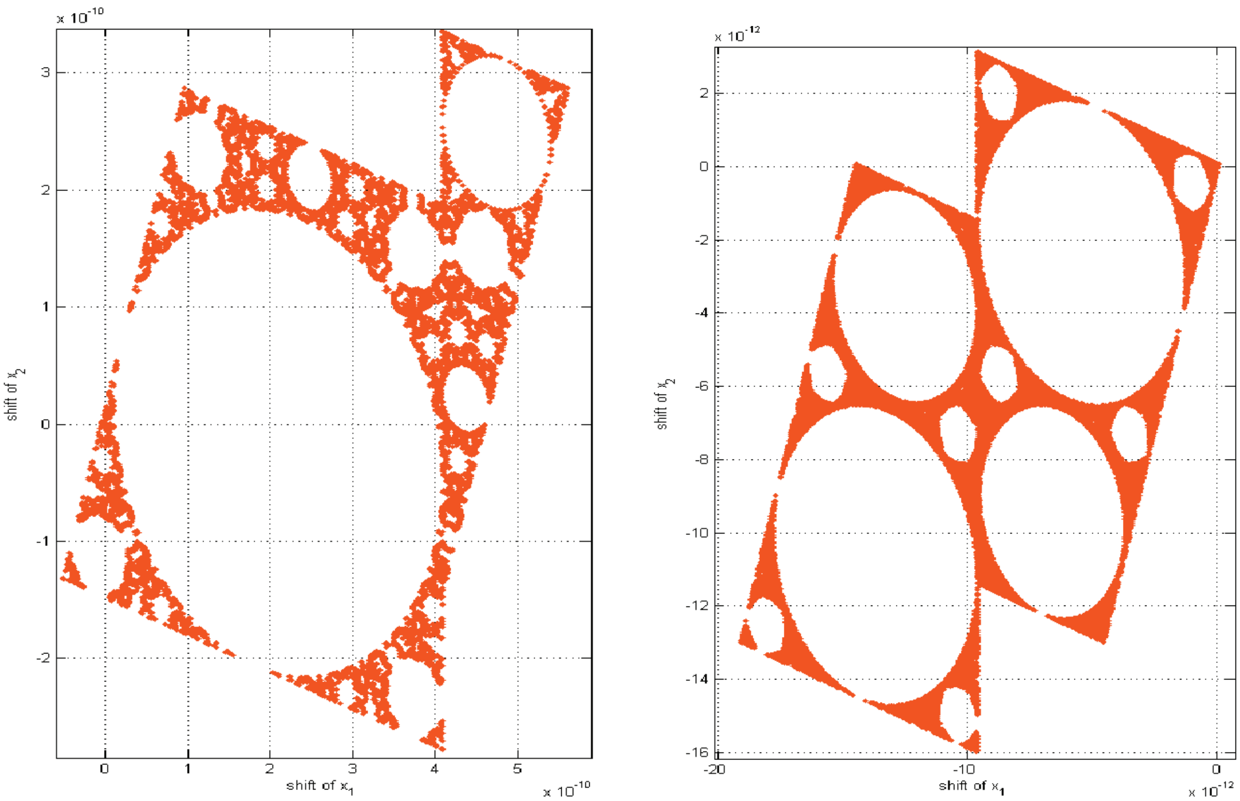


Fig. 1. The phase portraits of bandpass SDMs. (a) $N = 2$. (b) $N = 3$. (c) $N = 8$. The shifted phase portraits of bandpass SDMs. (d) $N = 16$. (e) $N = 32$. (f) $N = 37$.



(e) (f)

Fig. 1. (*Continued*)

occur for low bit quantizers. This result is also different from the existing results [Lin & Chua, 1991], in which the fractal behavior exists only for high bit quantizers.

Moreover, some interesting results are found. It is shown in Fig. 1 that the trajectories converge to a single ellipse when $N = 2$, while the trajectories exhibit elliptic fractal patterns when $N = 3$, $N = 8$, $N = 16$, $N = 32$ and $N = 37$. When the number of bits in the quantizers is increased by one, such as from $N = 2$ to $N = 3$, the trajectories change dramatically from a single ellipse to an elliptical fractal pattern. Besides, the regions of the elliptical fractal patterns get smaller and smaller as the number of bits of the quantizers is increased. This means, the amplitudes of the nonlinear oscillations are smaller when the number of bits of quantizers rises. Nevertheless, the frequency spectra are still very rich.

4. Conclusion

In this letter, some new results on multibit bandpass SDMs are presented. It is found that elliptical fractal patterns may be exhibited on the phase plane of multibit bandpass SDMs even though the saturation regions of the quantizers are not activated and high bit quantizers are used. In addition, we find that a bit change in the quantizers can change the behaviors of the systems dramatically.

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