NONLINEAR BEHAVIORS OF BANDPASS SIGMA DELTA MODULATORS WITH STABLE SYSTEM MATRICES

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ABSTRACT

It has been established that a class of bandpass sigma delta modulators (SDMs) may exhibit state space dynamics which are represented by elliptical or fractal patterns confined within trapezoidal regions when the system matrices are marginally stable. In this paper, it is found that fractal patterns may also be exhibited in the phase plane when the system matrices are strictly stable. This occurs when the sets of initial conditions corresponding to convergent or limit cycle behavior do not cover the whole phase plane. Based on the derived analytical results, some interesting results are found. If the bandpass SDM exhibits periodic output, then the period of the symbolic sequence must equal the limiting period of the state space variables. Second, if the state vector converges to some fixed points on the phase portrait, these fixed points do not depend directly on the initial conditions.

1. INTRODUCTION

Bandpass SDMs have many industrial and engineering applications because many systems are required to perform analog to digital conversions on bandpass signals [1]. By using bandpass SDMs, simple and relatively low precision analog components could achieve the objectives. Because of this advantage, this area draws much attention from the researchers in the community. Consequently, some methods for the analysis [3], [4] and design of bandpass SDMs have been proposed [2].

Since the quantization and feedback in bandpass SDMs introduces nonlinearities, limit cycles [3] and chaos [4] may occur. Some researchers utilize the nonlinear behavior in order to suppress unwanted tones from the quantizers [6]. The most common existing method is to place some unstable poles in the system matrices, so that chaotic behaviors will be exhibited in the systems, and the rich frequency spectra of these chaotic output signals break down the dominant oscillations at the outputs. However, by placing some unstable poles in the system matrices, the stability of the systems is degraded.

In the practical situation, there are leakages on the integrators [5]. This originates from the internal resistances of the components. Even though the leakages may sometimes be negligible, engineers and circuit designers may impose leakage on the integrators so as to improve the stability of the overall systems. Therefore, the eigenvalues of the system matrices are strictly inside the unit circle, and the system matrices are actually strictly stable.

Although there are some analytical results on the bandpass SDMs [4], most analysis is based on marginally stable system matrices only. For the bandpass SDMs with strictly stable system matrices, the existing results are primarily concerned with limit cycles, but not with fractal behavior. Intuitively, systems with stable system matrices will cause the trajectories to converge to some fixed points, and fractal behaviors would not occur. In this paper, we show that fractal behavior may also occur, and provide a justification for this and an analysis of its effect.

The organization of the paper is as follows. The analytical and simulation results of bandpass SDMs with strictly stable system matrices are given in Section II. Discussion and conclusion are given in Section III.

2. ANALYTICAL AND SIMULATION RESULTS

The bandpass SDMs in [7] with leakages can be modeled as follows:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{s}(k) + \mathbf{C}\mathbf{u}(k) \text{ for } k \geq 0 , \quad (1)$$

where $\mathbf{x}(k) \equiv [x_1(k) \quad x_2(k)]^T$ is the state vector function of the system, $\mathbf{u}(k) \equiv [u(k-2) \quad u(k-1)]^T$ is a vector containing the past two consecutive points from the input signal $u(k)$,

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 \\ -r^2 & 2r\cos q \end{bmatrix} \tag{2}$$

is the system matrix of the system, and

$$\mathbf{B} \equiv \mathbf{C} \equiv \begin{bmatrix} 0 & 0 \\ -r^2 & 2r\cos q \end{bmatrix}, \tag{3}$$

where **B** is the matrix associated with the nonlinearity, **C** is the matrix associated with the input, and

$$\mathbf{s}(k) \equiv [Q(x_1(k)) \quad Q(x_2(k))]^T \text{ for } k \geq 0\,, \tag{4}$$

in which the superscript ^{T} denotes the transpose operator,

$$Q(y) \equiv \begin{cases} 1 & y \geq 0 \\ -1 & otherwise \end{cases}, \tag{5}$$

$q \in (-p, p) \setminus \{0\}$ and $0 < r < 1$. As opposed to standard lowpass SDM systems, bandpass SDMs are designed to operate on high-frequency narrowband signals by shaping the noise from some frequency f_0 [4], where $f_0 = \frac{q\, f_s}{2p}$, in which f_s denotes the sampling frequency. At the desired frequency f_0, it has noise transfer function zero and signal transfer function one [4]. When $q \in \{-p, 0, p\}$, the system is either a lowpass SDM or a highpass SDM, which is out of the scope of the paper. The leakage of the system depends on the values of r. If r is closer to 0, then the poles are closer to the origin and the leakage is more serious. If r is closer to 1, then the poles are closer to the unit circle and the leakage is less significant. For an ideal bandpass SDMs, $r = 1$, the system reduces to that described in [7], and the system matrices are marginally stable. The value of $\mathbf{s}(k)$ can be viewed as symbols,

$$\mathbf{s}(k) \in \left\{ [1 \quad 1]^T, [1 \quad -1]^T, [-1 \quad 1]^T, [-1 \quad -1]^T \right\} \tag{6}$$

for $k \geq 0$, and $\mathbf{s}(k)$ is called a symbolic sequence.

In this paper, we only consider the cases when $\mathbf{x}(k)$ and $u(k)$ are real signals, that is $\mathbf{x}(k) \in \Re^2$ and $u(k) \in \Re$. We also assume that $u(k)$ is a constant input, that is $\mathbf{u}(k) = \mathbf{u}$ for $k \geq 0$.

2.1. Limit Cycle Behaviors

Define

$$\mathbf{D} \equiv \begin{bmatrix} re^{jq} & 0 \\ 0 & re^{-jq} \end{bmatrix} \tag{7}$$

and

$$\mathbf{T} \equiv \begin{bmatrix} \frac{1}{\sqrt{r}} e^{-\left(\frac{jq}{2}\right)} & \frac{1}{\sqrt{r}} e^{\frac{jq}{2}} \\ \sqrt{r} e^{\frac{jq}{2}} & \sqrt{r} e^{-\left(\frac{jq}{2}\right)} \end{bmatrix}. \tag{8}$$

Since $\mathbf{A}$ is a full rank matrix because $r \neq 0$, $\mathbf{A}$ can be decomposed via eigen decomposition. That is:

$$\mathbf{A} = \mathbf{TDT}^{-1}. \tag{9}$$

Let M be the period of the steady state of the output sequences (if it exists), that is

$$\mathbf{s}(k_0 + M + i) = \mathbf{s}(k_0 + i) \quad \forall i \geq 0\,, \tag{10}$$

in which $M \in Z^+$ and $k_0 \in Z^+ \cup \{0\}$. Define

$$\mathbf{x}_0^* \equiv \sum_{n=0}^{M-1} ?\, \mathbf{D}^{M-1-n} \left(\lim_{p \to +\infty} \sum_{m=0}^{p-1} \mathbf{D}^{mM} \right) \mathbf{T}^{-1} (\mathbf{Cu} - \mathbf{Bs}(k_0 + n)) \tag{11}$$

and

$$\mathbf{x}_i^* \equiv \mathbf{A}^i \mathbf{x}_0^* + \sum_{m=0}^{i-1} \mathbf{A}^{i-1-m} (\mathbf{Cu} - \mathbf{Bs}(k_0 + m)) \tag{12}$$

for $i = 1, 2, \cdots, M-1$. We have the following lemma.

Lemma 1

The following statements are equivalent:

i) $\mathbf{s}(k_0 + M + i) = \mathbf{s}(k_0 + i)$ $\forall i \geq 0$.

ii) $\lim_{k \to +\infty} \mathbf{x}(kM + k_0 + i) = \mathbf{x}_i^*$ for $i = 0, 1, \cdots, M-1$.

iii) $\mathbf{x}(0) \in \Xi_1 \equiv \{\mathbf{x}(0) : \exists k_0 \in Z^+ \cup \{0\}$ such that $\forall k \geq 0$, and $i = 0, 1, \cdots, M-1$, $Q(\mathbf{x}(kM + k_0 + i)) = Q(\mathbf{x}_i^*)\}$.

Proof:

For i) implies ii), from equation (1), we have: $\forall p, M \in ?^+$ and $\forall k \geq 0$,

$$\mathbf{x}(k + pM) = \mathbf{A}^{pM} \mathbf{x}(k) + \sum_{n=0}^{pM-1} \mathbf{A}^{pM-1-n} (\mathbf{Cu} - \mathbf{Bs}(k + n)). \tag{13}$$

From equation (9) and (i), we have:

$$\mathbf{x}(k_0 + pM) = ?\, \mathbf{D}^{pM} \mathbf{T}^{-1} \mathbf{x}(k_0) + \sum_{n=0}^{M-1} ?\, \mathbf{D}^{M-1-n} \left(\sum_{m=0}^{p-1} \mathbf{D}^{mM} \right) \mathbf{T}^{-1} (\mathbf{Cu} - \mathbf{Bs}(k_0 + n)). \tag{14}$$

Hence, we have:

$$\lim_{p \to +\infty} \mathbf{x}(k_0 + pM) = \mathbf{x}_0^*. \tag{15}$$

By substituting equation (15) into equation (1), the result follows directly.

For ii) implies i), since

$$\lim_{k \to +\infty} \mathbf{x}(kM + k_0 + i) = \mathbf{x}_i^* \text{ for } i = 0, 1, \cdots, M-1, \tag{16}$$

then $\exists k_1 \geq 0$ such that

$$Q(\mathbf{x}(kM + k_0 + i)) = Q(\mathbf{x}_i^*) \tag{17}$$

for $k \geq k_1$ and $i = 0, 1, \cdots, M-1$. Hence, the result follows directly.

For ii) implies iii), since

$$\lim_{k \to +\infty} \mathbf{x}(kM + k_0 + i) = \mathbf{x}_i^* \text{ for } i = 0, 1, \cdots, M-1, \tag{18}$$

then $\exists k_1 \geq 0$ such that

$$Q(\mathbf{x}(kM + k_0 + i)) = Q(\mathbf{x}_i^*) \tag{19}$$

for $k \geq k_1$ and $i = 0, 1, \cdots, M-1$. Hence, the result follows directly.

For iii) implies i), since

$$Q(\mathbf{x}(kM + k_0 + i)) = Q(\mathbf{x}_i^*) \tag{20}$$

for $k \geq 0$ and for $i = 0, 1, \cdots, M-1$, the result follows directly.

This completes the whole proof of the lemma. ■

Lemma 1 associates the steady state of periodic output with a specific set of initial conditions and a corresponding dynamical behavior of the system. According to Lemma 1, we can easily see that the trajectories will converge to the set of fixed points $\{\mathbf{x}_0^*, \mathbf{x}_1^*, \cdots, \mathbf{x}_{M-1}^*\}$, and the periodicity of the steady states of the output sequence is equal to the number of fixed points on the phase plane. That implies that all the fixed points (more than or equal to 2) cannot be in the same quadrant. For example, if $M = 2$, then there are two fixed points on the phase plane and these two fixed points are located in different quadrants.

The significance of Lemma 1 is that it provides useful information for estimating the periodicity of the steady state of output sequences via the phase portrait. Moreover, Lemma 1 provides useful information to the SDM designers to determine the set of initial conditions which leads to limit cycle behavior.

It is worth noting that although the state vector is converging to a periodic orbit, it never reaches these periodic points. That means, the state vector is aperiodic even though the output sequence is eventually periodic. This result is different from the case when $r = 1$ and q is a rational multiple of p.

Moreover, although $\mathbf{x}_i^*$, for $i = 1, 2, \cdots, M-1$, depends on $\mathbf{s}(i)$, for $i = 1, 2, \cdots, M-1$, it does not depend on $\mathbf{x}(0)$ directly. That is, the fixed points leading to a given symbol sequence are not directly depended on the initial conditions.

When $M = 1$, the output sequence will become constant and there is only one single fixed point on the phase portrait. The trajectory will converge to this fixed point, denoted as $\mathbf{x}^*$. The significance of this result is that it allows SDM designers to determine the set of initial conditions so that limit cycle behavior is *avoided*.

It is worth noting that the state vectors of the corresponding linear system will converge to $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{Cu}$, which is not the same as that of $\mathbf{x}^*$. Comparing these two values, there are DC shifts and the DC shifts are exactly dropped at the output sequences, that is:

$$(\mathbf{I}-\mathbf{A})^{-1}\mathbf{Cu} - \mathbf{x}^* = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{Bs}_{k_0}, \qquad (21)$$

in which

$$\mathbf{s}(k) = \mathbf{s}_{k_0} \text{ for } k \geq k_0. \qquad (22)$$

In addition, this phenomenon is quite different from the case of lowpass SDMs. In such a situation, the average output sequence will approximate the input values even though limit cycle behavior occurs.

Although the nonlinearity is always activated, the rate of convergence only depends on r when the output sequence becomes steady. This is because the DC terms do not affect the rate of convergence. However, if we look at the transient response of the system, that is, the time duration when the output sequence is not constant, the system dynamics could be very complex.

Figure 1 shows the response of the state variables of a bandpass SDM with

$$r = 0.9999, \quad q = \cos^{-1}(-0.158532), \quad \mathbf{u} = -0.3[1 \quad 1]^T$$
and $$\mathbf{x}(0) = [0 \quad 0.5]^T. \qquad (23)$$

The state variables will converge to the same fixed value and the output sequences will become constant for $k \geq 2154$.

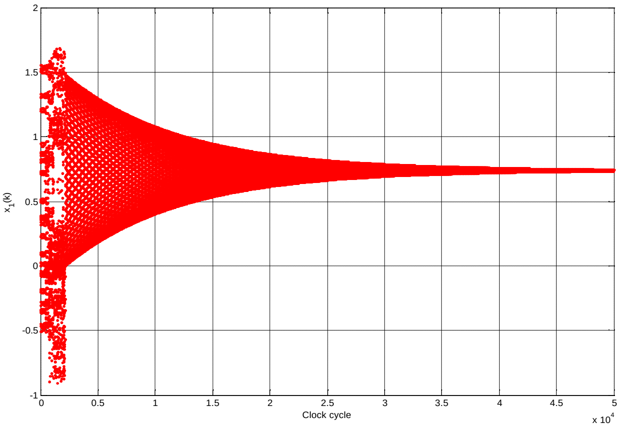

Figure 1. The state variable $x_1(k)$.

Figure 2 shows the state trajectory of a bandpass SDM with

$$r = 0.99, \quad q = \cos^{-1}(-0.158532), \quad \mathbf{u} = -0.3[1 \quad 1]^T \text{ and}$$
$$\mathbf{x}(0) = [0 \quad 0.5]^T. \qquad (24)$$

The state trajectory will converge to two fixed points and the output sequences are periodic with period 2 for $k \geq 3$.

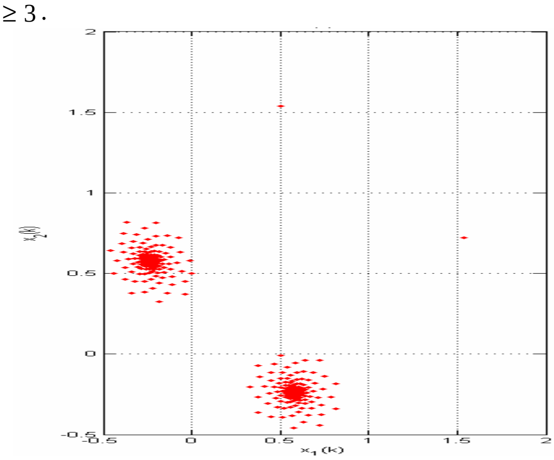

Figure 2. The phase portrait when $M = 2$.

2.2. Fractal Behaviors

Intuitively, fractals would not be exhibited on the phase plane when the system matrices of the bandpass SDM are strictly stable. However, if

$$\Xi_2 \equiv \Re^2 \setminus \Xi_1 \neq \Phi, \qquad (25)$$

in which Φ denotes the empty set, then there exists some initial conditions which would not result in the convergence of the periodic output sequences. Since $\forall \mathbf{x}(0) \in \Xi_2$, there does not exist $k_0 \in Z^+ \cup \{0\}$ such that $\mathbf{x}(k_0) \in \Xi_1$, the region Ξ_1 in the phase plane has to be empty. As a result, a fractal pattern would be exhibited on the phase plane. As the output sequences corresponding to limit cycle behaviors are eventually periodic, the output sequences for $\forall \mathbf{x}(0) \in \Xi_2$ are aperiodic.

Figure 3 shows the state trajectory of a bandpass SDM with

$r = 1 - 10^{-6}$, $q = \cos^{-1}(-0.158532)$, $\mathbf{u} = -0.3[1 \quad 1]^T$ and $\mathbf{x}(0) = [0 \quad 0.5]^T$. (26)

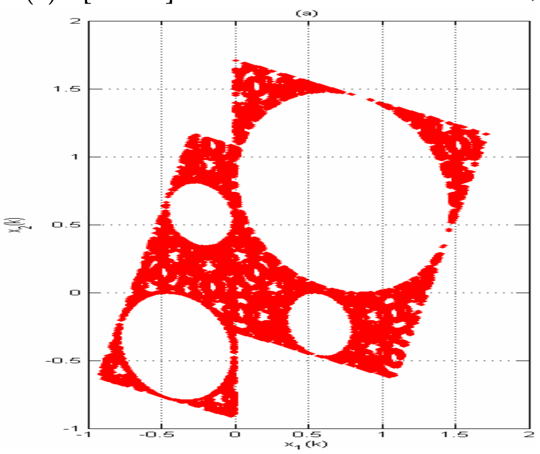

Figure 3. The phase portrait when output sequences are aperiodic.

It can be seen from the figure that fractal pattern is exhibited on the phase plane and the trajectories neither converge to the boundaries of the trapezoids nor any fixed points in the phase portrait. Measurements of the fractal dimension [8] are estimated at 1.78 for the box counting dimension, 1.75 for the information dimension, and 1.72 for the correlation dimension.

One possible implication of the results obtained in the paper is that it is not necessary to place unstable poles in the system matrices of bandpass SDMs to generate signals with rich frequency spectra in order to suppress unwanted tones from quantizers. It is shown in this paper that fractals can be generated via system matrices with strictly stable poles. Since the output sequences are aperiodic, which consist of rich frequency spectra, the unwanted tones could be suppressed using these aperiodic signals without the tradeoff of the stability of the systems.

3. DISCUSSION AND CONCLUSION

In this paper, we account for the occurrence of fractal patterns for bandpass SDMs with strictly stable system matrices. If the sets of initial conditions corresponding to the eventually periodic output do not cover the whole phase plane, then fractal patterns would be exhibited. Some interesting results are found. First, for a periodic output sequence, the limiting period of the state space variables must equal the period of the symbolic sequence. This implies that all the periodic points cannot be in the same quadrant. If the state vector converges to some fixed points on the phase portrait, these fixed points do not depend on the initial condition directly.

One implication of the results obtained in this paper is that we can generate signals with rich frequency spectra by using strictly stable system matrices in order to suppress unwanted tones generated by the quantizers. Thus limit cycles may be avoided without a tradeoff in the stability of the bandpass SDM.

4. ACKNOWLEDGEMENT

The work obtained in this paper was supported by a research grant from Queen Mary, University of London.

5. REFERENCES

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