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Towards a procedure for stability analysis of high order sigma delta modulators

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ABSTRACT

One of the greatest unsolved problems in the theory of sigma delta modulation concerns the ability to analytically derive the stability, or boundedness, of a high order sigma delta modulator (SDM). In this work, we describe the existing literature and try to clarify the issues involved. We fully derive the stability of first order sigma delta modulators, and derive some important results for the basic second order sigma delta modulator. For third order sigma delta modulators, we describe interesting simulated results as well as sketch a proof of instability, based on linear programming, for one particular SDM. Finally, we present two theoretical results concerning stability of general high order SDMs that point towards promising directions of future research.

1. INTRODUCTION

The stability question may be phrased in many different ways. At its core, we would like to derive the value of constant input such that, for initial conditions set to zero, the magnitude of the state space variables will diverge towards infinity. A similar question is, given initial conditions and a constant input value, we should be able to determine if this leads to stable behavior. Other important problems are the determination of the invariant set, trapping region, or basin of attraction. These are related concepts which all, in some sense, refer to the set of state space values which lead to a bounded solution. The size and location of this set should give some indication of the stability as well.

Our goal is to derive an analytical method of determining the stability of sigma delta modulators. That is, we wish to present a mathematical framework, based on state space modeling and symbolic dynamics, for deriving the stability of 1-bit Sigma Delta Modulators (SDMs). We will focus on DC inputs, since this represents the most relevant (and easiest) practical condition. We wish to determine the maximum value of the constant applied input which will produce stable output indefinitely.

We are further concerned with the dependence of stability on initial conditions of the state variables. For unstable input values, the time until instability is reached, and the effect of clipping to enforce stability, are both of primary concern. Understanding of the effects of dithering, small disturbances of the state space variables, and the use of nonconstant input, is also necessary. A justification for the use of resonator sections to improve stability is also unknown. Finally, examples using high quality, high order feedback and feedforward designs are necessary to demonstrate any theoretical results.

This work does not claim to have achieved any of the above mentioned tasks. Rather, we feel it was necessary to describe the issues thoroughly and formally, as well as the existing approaches and results on low order SDMs, and to indicate the most promising directions of future research.

2. PRIOR WORK

2.1. Traditional stability analysis

Almost all SDM designs may be characterized as piecewise linear maps (piecewise affine maps, to be exact). This would imply that a first approach would be to use standard Lyapunov stability theory, except as applied to maps rather than piecewise continuous systems. Unfortunately, most seemingly related work has dealt with a different definition of stability. Feng's recent work[1] on "Stability Analysis of Piecewise Discrete-Time Linear Systems" was concerned with a global exponential stability. That is, all solutions will tend to the origin eventually. This is the typical problem in Lyapunov stability theory and thus not directly applicable to the stability issues we are concerned with. Kantner[2], though working on similar asymptotic stability problems, also made the observation that linear programming techniques can be applied to related stability issues.

2.2. Computational approaches to SDM stability

Researchers experienced with sigma delta modulation have, in general, avoided the Lyapunov stability approach. Risbo[3] discussed stability of SDMs in detail, primarily from a nonlinear dynamics perspective. But, with the exception of first order SDMs, he did not attempt a method for its determination. However, he introduced some important concepts such as boundary crises and escape routes. Much

A computational approach to finding the invariant sets is derived by Schreier[4-6]. Although neither analytical nor rigorous, it is significant because source code is available, and because results are provided which may be confirmed or denied by other methods.

2.3. Analytical approaches to SDM stability

To the best of our knowledge, there is no successful analytical approach to stability in high order SDMs (greater than 3). There are several alternative approaches to stability in second order SDMs, some preliminary work on third order designs, and only 'sketched' approaches to stability in higher order SDMs. Thus the question becomes, "Can any existing approaches be extended to higher order SDMs,?" Of course, there is the related question of whether existing approaches are correct.

Hein and Zakhor's approach[7] is to use the limit cycles as a measure of stability. Their method is not rigorous (and in some sense not analytical) in that it postulates that the limit cycles have a convergent bound on the state space variables, and that this is also the bound for non-limit cycle behaviour. Our own research has found limit cycles which contradict this[8]. However, the method seems to work and the results agree well with those of Farrell and Feely[9].

Wang[10] used an interesting approach. He converted a third order modulator to a continuous time system by looking at the vector field equations. Then, by considering only boundary points, he is able to convert the 3 dimensional flow into a 2 dimensional return map. Fixed points of this map then yield insight into stability of the SDM. This is a very complicated method (although the math does not become intractable) and it is unclear if it may be extended to higher order modulators and if it is fully justified.

Zhang[11, 12] uses a model of the quantiser to estimate stability of a third order SDM. Though this seems to work, the linearization implies that important phenomena have been omitted. Furthermore, there is little comparison of their results with simulation. Another work by Zhang[13] bears a strong resemblance to the linear programming approach of Feely, though it seems oversimplistic.

Steiner and Yang[14, 15] use a transformation which decouples the state space variables except through their interaction in the quantization function. They suggest how this may be used to tackle stability but there is little actual analysis. This approach has been expanded by Wong[16] to deal with practical high order SDMs. However, the analysis appears nearly intractable. Wong provides simulated results for many high order SDMs, but his analysis does not seem to confirm simulation.

A straight linear programming approach is used by Farrell and Feely[9]. This successfully finds the bounds on the second order SDM and may be extended to second order SDMs with leaky or chaotic integrators. The math appears correct and tractable, and their results bear stronger agreement with simulation than the results of Hein and Zakhor.

The remainder of this document is concerned with some initial results on stability. Section 3 provides some easy proofs for first order modulators. Section 4 outlines Farrell and Feely's method for computing the stability and bounds of a second order SDM and puts this technique into a form such that it may be applied to more general SDM designs. Section 5 provides several interesting simulated results of the stability of generic third order feedforward and feedback SDM designs, as well as a sketched proof of the instability of the basic third order SDM. Section 6 provides 2 results for arbitrary 1 bit SDMs; that they will always be unstable for input magnitude greater than 1, and that the output oscillates between positive and negative values, even when unstable, for input less than 1. Notably, Figure 4 demonstrates this phenomenon by using a log-log plot to depict the output of an unstable SDM. This section concludes with a description of how linear programming may be applied to the stability of high order SDMs.

3. FIRST ORDER SDMS

3.1. Proof that the output is bounded for a first order SDM with -1<*U*<1

A first order SDM is given by

$$s^{(n+1)} = s^{(n)} + u - y^{(n)} \quad (1)$$

We assume that the input is bounded by -1<*u*<1 (this assumption is justified in the following section, 3.2). Then the following 2 relationships show that a negative initial *s* will increase until it is positive, and a positive *s* will decrease until it is negative.

$$\begin{aligned} s^{(n)} < 0 &\Rightarrow s^{(n+1)} = s^{(n)} + u + 1 > s^{(n)} \\ s^{(n)} \geq 0 &\Rightarrow s^{(n+1)} = s^{(n)} + u - 1 < s^{(n)} \end{aligned} \quad (2)$$

Thus it is oscillating between positive and negative values.

We want to know, assuming that atleast one bit flip has occurred (i.e., the transient behavior has passed and we are not starting from *arbitrary* initial conditions), what is the range of values which *s* can take.

Note from (2) that the maximum value of *s* occurs when the previous value is just below zero and the minimum value occurs when the previous value is equal to zero

$$\begin{aligned} s^{(n)} \underset{<}{\to} 0 &\Rightarrow s^{(n+1)} \sim u + 1 \\ s^{(n)} = 0 &\Rightarrow s^{(n+1)} = u - 1 \end{aligned} \quad (3)$$

Thus *s* is limited to the range [-1+*u*;1+*u*].

3.2. Maximum # of consecutive equal output bits for a first order SDM

Eq. (1) can be iterated to give, when all output bits are assumed positive,

$$s^{(n+N)} = s^{(n)} + N(u-1) \quad (4)$$

We know that the maximum value of $s^{(n)}$ is *u*+1, so from Eq. (4), the maximum number of positive output bits, $N^{+}_{\max}$, is given by the smallest *N* such that

$$N \leq \frac{1+u}{1-u} + 1 = \frac{2}{1-u} \quad (5)$$

Note that, this also gives the stability limits, since

$$u \to +1 \Rightarrow N^{+}_{\max} \to \infty \quad (6)$$

Furthermore, negative input results in only isolated positive output bits.

$$-1 < u < 0 \Rightarrow N^{+}_{\max} = 1 \quad (7)$$

Similarly, the maximum number of negative output bits, $N^{-}_{\max}$, is given by the smallest *N* such that

$$N \leq \frac{2}{1+u} \quad (8)$$

and

$$\begin{aligned} &u \to -1 \Rightarrow N^{-}_{\max} \to \infty \\ &0 < u < 1 \Rightarrow N^{-}_{\max} = 1 \end{aligned} \quad (9)$$

3.3. Derivation of the stability range for a chaotic or leaky 1st order SDM

A first order SDM with a chaotic or leaky integrator is given by

$$s^{(n+1)} = cs^{(n)} + u - y^{(n)} \quad (10)$$

where *c* is positive (typically close to 1). This can be iterated to give, when the output bits all have the same sign,

$$s^{(n+N)} = c^{N} s^{(n)} + (u-y)[\sum_{i=1}^{N} c^{i-1}] \quad (11)$$

The boundaries are the same as for the ideal 1st order SDM,

$$\begin{aligned} y^{(0)} = -1, y^{(1)} = +1 &\to 0 < s^{(1)} < u + 1 \\ y^{(0)} = +1, y^{(1)} = -1 &\to u - 1 < s^{(1)} < 0 \end{aligned} \quad (12)$$

The change in value of *s* between iterations is given by

$$\begin{aligned} \Delta s &= s^{(n+1)} - s^{(n)} \\ &= (c-1)s^{(n)} + u - y^{(n)} \end{aligned} \quad (13)$$

For the input to be stable, the minimum change must be negative when there is a positive output bit, and the maximum change must be positive when there is a negative output bit.

From (12),

$$(c-1)(u+1)+u-1<0$$
$$(c-1)(u-1)+u+1>0 \quad (14)$$

by solving for u, we obtain the maximum and minimum inputs for stability

$$1-2/c<u<2/c-1 \quad (15)$$

4. DERIVATION OF THE BOUNDS FOR A SECOND ORDER FEEDBACK SDM

4.1. Method

This proof of the stability of a standard second order SDM is based on the method outlined in [9]. Here, we have rephrased the results into the preferred terminology and elaborated and commented on several parts.

This proof has several steps. First we assume that there have been some number N^- iterations with negative output. We can then identify the maximum values of the state space variables for the first positive output bit. We use this value to identify the maximum number of positive output bits which results, N^+. We can then find the maximum number of negative output bits which result from the N^+ positive bits. This new value of N^- is strictly less than N^+ and hence the oscillations are bounded.

Note that N here refers to the number of iterations, not SDM order, and that the equations for the SDM, though standard, are not the same as those used in some other sources.

The equations of the standard 2nd order FB design are given by

$$s_1^{(n+1)}=s_1^{(n)}+u-y^{(n)}$$
$$s_2^{(n+1)}=s_2^{(n)}+s_1^{(n+1)}-y^{(n)} \quad (16)$$

where $y^{(n)}=\operatorname{sgn}(s_2^{(n)})$ and we assume constant input

4.2. Maximum values of the state space variables after N^- negative iterations

If there are N iterations without the quantiser changing sign, $y^{(n)}=...=y^{(n+N-1)}=y$ then the following equations give the resultant dynamics

$$s_1^{(n+N)}=s_1^{(n)}+N(u-y)$$
$$s_2^{(n+N)}=s_2^{(n)}+\frac{N(N+1)}{2}(u-y)+N(s_1^{(n)}-y) \quad (17)$$

Since we use them later, the inversion of this is

$$s_1^{(n)}=s_1^{(n+N)}-N(u-y)$$
$$s_2^{(n)}=s_2^{(n+N)}+\frac{N(N-1)}{2}(u-y)-N(s_1^{(n+N)}-y) \quad (18)$$

We suppose there are exactly N^- consecutive -1 bits. That is, the bitstream is

$$\underbrace{+1}_{Q(0)},\underbrace{-1,-1,...-1,-1}_{Q(1),...Q(N^-)},\ \underbrace{+1}_{Q(N^-+1)} \quad (19)$$

This gives N^-+2 constraints on the state space variables.

$$s_2^{(1)}\ge s_1^{(1)}-1$$
$$s_2^{(1)}<0$$
$$s_2^{(2)}<0$$
$$...$$
$$s_2^{(N^-)}<0$$
$$s_2^{(N^-+1)}\ge 0 \quad (20)$$

where the first condition is derived from

$$s_2^{(0)}\ge 0\Rightarrow s_2^{(1)}=s_2^{(0)}+s_1^{(1)}-y^{(0)}\ge s_1^{(1)}-1 \quad (21)$$

We wish to find the maximum value of $s_1^{(N^-+1)}$. That is, given N^- consecutive -1 output bits, what is an upper bound on where the first state space variable will be located for a +1 output bit.

Rewriting each of these as a constraint on the N+1 state, we have, from (18),

$$s_2^{(N^-+1)}\ge[N^-+1]s_1^{(N^-+1)}+N^--1-(N^-+1)N^-(u+1)/2$$
$$s_2^{(N^-+1)}<N^-(s_1^{(N^-+1)}+1)-N^-(N^--1)(u+1)/2$$
$$s_2^{(N^-+1)}<[N^--1](s_1^{(N^-+1)}+1)-(N^--1)(N^--2)(u+1)/2$$
$$...$$
$$s_2^{(N^-+1)}<s_1^{(N^-+1)}+1$$
$$s_2^{(N^-+1)}\ge 0$$

(22)

This is a linear programming problem, except that the number of constraints is unknown. If we consider just a subset of these conditions and ignore the importance of other constraints,

$$a)s_2^{(N^-+1)}\ge[N^-+1]s_1^{(N^-+1)}+N^--1-(N^-+1)N^-(u+1)/2$$
$$b)s_2^{(N^-+1)}<s_1^{(N^-+1)}+1$$
$$c)s_2^{(N^-+1)}\ge 0$$

(23)

The inequalities in Eq. (23) are depicted in Figure 1, for $N^-=3$ and $u=0$. One can see that an upper bound on s_1 occurs when both the first two conditions in (23) become equalities. Solving for s_1 then gives

$$s_{1\max}=2/N^--1+(N^-+1)(u+1)/2 \quad (24)$$

Of course, it is conceivable that the other constraints will create an even stricter upper bound, especially for unusual values of the parameters. But for now, we will use just this constraint.

From the second condition in Eq. (23), we also have an upper bound on s_2,

$$s_{2\max} = 2/N^- + (N^- + 1)(u+1)/2 \quad (25)$$

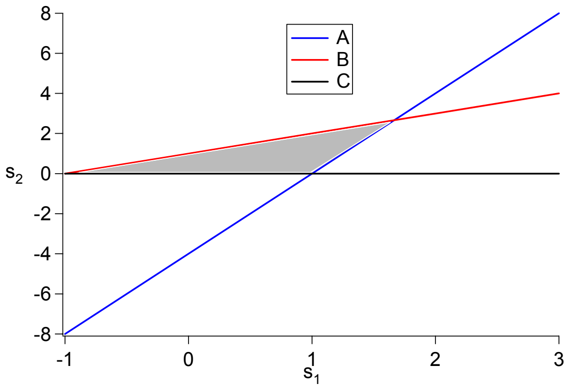

Figure 1. The inequalities of Eq. (23). The shaded region represents the allowable solutions.

4.3. Maximum number of resultant iterations with positive output

Now we suppose there are exactly N^- consecutive -1 bits, followed by N^+ consecutive +1 bits. That is, the bitstream is

$$\underbrace{+1}_{y(0)}, \underbrace{-1,-1,\ldots -1,-1}_{y(1),\ldots y(N^-)}, \underbrace{+1,+1,\ldots +1,+1}_{y(N^-+1),\ldots y(N^-+N^+)} \quad (26)$$

To find the maximum number of iterations that will then occur with positive output, we need to find the smallest integer value of N^+ such that

$$s_2^{(N^- + N^+ + 1)} < 0 \quad (27)$$

where, we choose the largest values of the state space variables, given by (24) and (25), since this value is furthest from negative output.

$$s_2^{(N^-+1)} = s_{2\max} = s_{1\max} + 1 \quad (28)$$

from (17) and (28), for y=1,

$$s_2^{(N^-+N^++1)} = s_{1\max} + 1 + \frac{N^+(N^+ + 1)}{2}(u-1) + N^+(s_{1\max} - 1) \quad (29)$$

So, an upper bound on the number of positive iterations following N^- negative iterations is given by N^+ where N^+ is the smallest integer such that,

$$[N^+ + 1]s_{1\max} - N^+ + 1 + \frac{N^+(N^+ + 1)}{2}(u-1) < 0 \quad (30)$$

or

$$[N^+ + 1][2/N^- - 1 + (N^- + 1)(u+1)/2] - N^+ + 1 + \frac{N^+(N^+ + 1)}{2}(u-1) < 0 \quad (31)$$

For u=0, Table 1 gives the value of the left hand side of (30) for different values of N^- and N^+ between 1 and 7. From this, we can derive Table 2, which gives the maximum number of resultant positive iterations for a given number of negative iterations.

Table 1. Values of the condition for the upper bound on N^+, from (30) as a function of N^- and N^+.

		N^+						
		1	2	3	4	5	6	7
N^-	1	3	2	0	-3	-7	-12	-18
	2	2	1/2	-2	-11/2	-10	-31/2	-22
	3	7/3	1	-4/3	-14/3	-9	-43/3	-62/3
	4	3	2	0	-3	-7	-12	-18
	5	19/5	16/5	8/5	-1	-23/5	-46/5	-74/5
	6	14/3	9/2	10/3	7/6	-2	-37/6	-34/3
	7	39/7	41/7	36/7	24/7	5/7	-3	-54/7

We can see that for $N^->4$, the resulting N^+ must be strictly less than N^-. An identical proof shows that, if we consider positive output preceding negative output, then for $N^+>4$, the resulting N^- is strictly less than N^+. Thus u=0 gives a globally stable solution.

Table 2. The upper bound on N^+ as a function of N^- .

N^-	$N^+_{\max}$
1	4
2	3
3	3
4	4
5	4
6	5
7	6

5. THIRD ORDER SDMS

5.1. Simulated Results

Consider the system

$$\begin{aligned} s_1^{(n+1)} &= s_1^{(n)} + c_1 u - c_1 y^{(n)} \\ s_2^{(n+1)} &= s_2^{(n)} + s_1^{(n)} - c_2 y^{(n)} \\ s_3^{(n+1)} &= s_3^{(n)} + s_2^{(n)} - y^{(n)} \end{aligned} \quad (32)$$

and the system

$$\begin{aligned} s_1^{(n+1)} &= s_1^{(n)} + c_1 u - c_1 y^{(n)} \\ s_2^{(n+1)} &= s_2^{(n)} + s_1^{(n+1)} - c_2 y^{(n)} \\ s_3^{(n+1)} &= s_3^{(n)} + s_2^{(n+1)} - y^{(n)} \end{aligned} \quad (33)$$

Both Eq. (32) and (33) represent third order SDMs with only 2 coefficients, c_1 and c_2. We are concerned with finding values of the coefficients which yield stable behavior. Via simulation, the stable region was found to be given by the shaded regions in Figure 2(a) and Figure 2(b) for Eq. (32) and (33) respectively.

We note that this gives simple constraints for the coefficients in both cases. For Eq. (32), we have

$$\begin{aligned} c_2 &\le 2c_1 \\ c_2 &\le 1 - c_1 \end{aligned} \tag{34}$$

and for Eq. (33), we have the simple constraint,

$$c_1 \le 1/2 \tag{35}$$

It should be possible to verify both of these relationships, and this is a current direction of research.

If we consider a third order feedforward design, then the dynamics are somewhat different. This design is given by

$$\begin{aligned} s_1^{(n+1)} &= s_1^{(n)} + u - y^{(n)} \\ s_2^{(n+1)} &= s_2^{(n)} + s_1^{(n)} \\ s_3^{(n+1)} &= s_3^{(n)} + s_2^{(n)} \end{aligned} \tag{36}$$

where

$$y^{(n)} = Q(s_1^{(n)} + c_2 s_2^{(n)} + c_3 s_3^{(n)}) \tag{37}$$

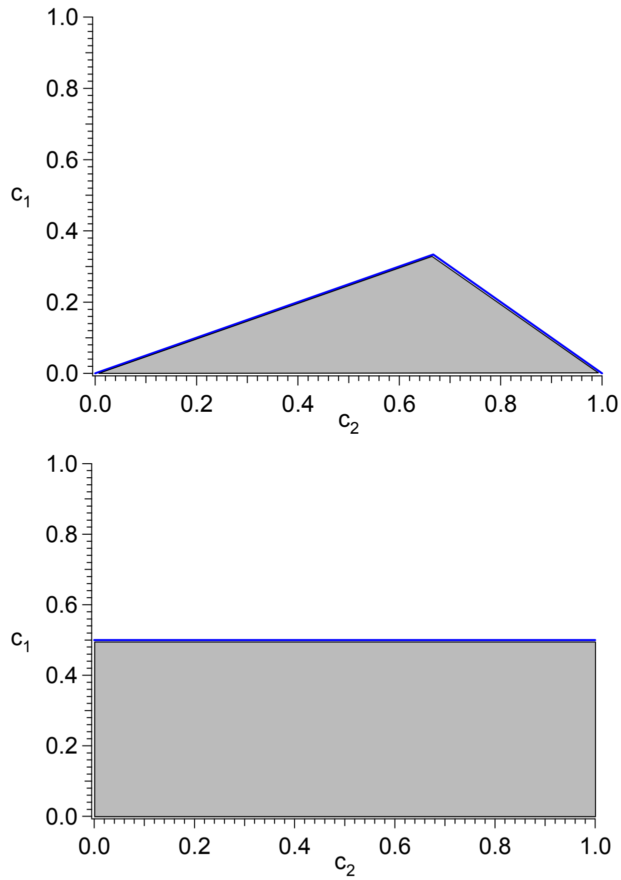

Figure 2. (a) Stable region of a 3rd order SDM from Eq. (32), as a function of the coefficients, for zero input and initial conditions. (b) Stable region, under the same constraints, for an SDM given by Eq. (33).

The stable region is depicted in Figure 3, for zero initial conditions, and for inputs 0 (a) and 0.5 (b). One can see that the region is more complicated than the simple regions given for the feedback designs. However, there still appears to be some relationships between coefficients and stability. For instance, in Figure 3(a), the line

$$c_2 \le 2c_3 \tag{38}$$

Determines most stable solutions for $0 \le c_3 \le 0.75$.

5.2. Proof of unboundedness of a certain 3rd *feedback* order SDM

This is the standard 3rd order FB design. We assume no input, and we wish to show that this yields unstable behavior

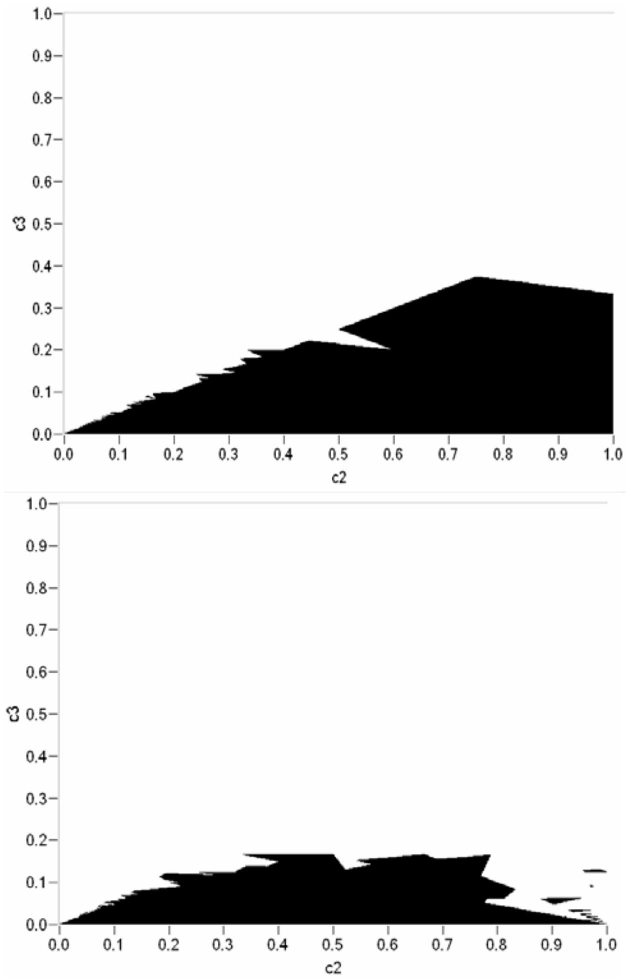

Figure 3. (a) Stable region of a 3rd order SDM from Eq. (36), as a function of the coefficients, for zero input and zero initial conditions. (b) Stable regions for the same SDM, but with constant 0.5 input.

The equations are given by

$$\begin{aligned} s_1^{(1)} &= s_1^{(0)} + u - y^{(0)} \\ s_2^{(1)} &= s_2^{(0)} + s_1^{(1)} - y^{(0)} \\ s_3^{(1)} &= s_3^{(0)} + s_2^{(1)} - y^{(0)} \end{aligned} \quad (39)$$

where $y^{(n)} = \mathrm{sgn}(s_3^{(n)})$. This yields

$$\begin{aligned} s_1^{(n)} &= s_1^{(0)} + nu - \sum_{i=0}^{n-1} y^{(i)} \\ s_2^{(n)} &= s_2^{(0)} + ns_1^{(0)} + n(n+1)u/2 - \sum_{i=0}^{n-1}(n+1-i)y^{(i)} \\ s_3^{(n)} &= s_3^{(0)} + ns_2^{(0)} + n(n+1)s_1^{(0)}/2 \\ &+ n(n+1)(n+2)u/6 - \sum_{i=0}^{n-1}(n+1-i)(n+2-i)y^{(i)}/2 \end{aligned} \quad (40)$$

So, for a given sequence, $y(0), y(1), ...y(N-1)$, we have the set of constraints

$$y^{(i)} s_3^{(i)} \geq 0 \quad i = 0...n-1 \quad (41)$$

We would like to maximize u such that these constraints hold. This is a linear programming problem. In this case, using linear programming (Numerical Recipes implementation), we find that, for $u \geq 0$ there is no feasible solution for the sequence

$$\underbrace{-1}_{y(0)}, \underbrace{+1, +1, ...+1, +1}_{y(1),...y(N^+)}, \underbrace{-1, -1, ...-1, -1}_{y(N^+ +1),...y(N^- + N^+)}, \underbrace{+1}_{y(N^- + N^+ + 1)} \quad (42)$$

where $N^+ \leq N^-$ except where N^+ is 1 or 2. However, if we consider,

$$\underbrace{+1}_{y(0)}, \underbrace{-1, -1, ...-1, -1}_{y(1),...y(N^-)}, \underbrace{+1, +1, ...+1, +1}_{y(N^- +1),...y(N^- + N^+)}, \underbrace{-1}_{y(N^- + N^+ + 1)} \quad (43)$$

and N^- is 1 or 2, then there are no solutions for $N^- \leq N^+$. This implies that the length of a sequence of +1s or -1s is growing. We can thus conclude that this system is unbounded.

6. RESULTS FOR ARBITRARY ORDER SDMS

An arbitrary order feedforward SDM may be represented as,

$$\begin{aligned} s_1^{(n+1)} &= s_1^{(n)} + u - y^{(n)} \\ s_2^{(n+1)} &= s_1^{(n)} + s_2^{(n)} \\ &... \\ s_N^{(n+1)} &= s_{N-1}^{(n)} + s_N^{(n)} \end{aligned} \quad (44)$$

where $y^{(n)} = \mathrm{sgn}(\mathbf{c} \cdot \mathbf{s})$ and we have assumed constant input u.

Here, we will show that the state space variables are *always* unbounded for $|u|>1$, and that the state space variables oscillate between positive and negative values for $|u|<1$. Note that this oscillation does not guarantee stable behavior, but an understanding of the oscillations may lead to an understanding of stability.

6.1. Proof that the output is unbounded for any FF SDM with constant input >1.

From (44), it is easy to see that

$$\begin{aligned} s_i^{(n)} > 0 &\Rightarrow s_{i+1}^{(n+1)} > s_{i+1}^{(n)} \\ s_i^{(n)} < 0 &\Rightarrow s_{i+1}^{(n+1)} < s_{i+1}^{(n)} \end{aligned} \quad (45)$$

Assume u>1. So $u - y^{(n)} > 0$, regardless of **c**.

Therefore,

$$s_1^{(n+1)} > s_1^{(n)} \quad (46)$$

e.g., s_1 always increases. Hence, for some k, $s_1^{(k)} > 0$.

At which point s_2 will increase, and at some point it will become positive, and so on. This implies that, at some point all state space variables will increase.
Similarly, if u<1, at some point, all state space variables will decrease.
Thus, for any feedforward SDM in Jordan form, the bounds are always <=1.

6.2. Proof that the output oscillates for SDMs with -1<*U*<1.

Assume 0<u<1, and $y^{(n)}$>0
So

$$s_1^{(n+1)} = s_1^{(n)} + u - 1 < s_1^{(n)} \qquad (47)$$

So s_1 decreases. s_2 may still increase, but eventually s_1 becomes less than 0. Then s_2 starts to decrease, and so on. Eventually $s_N^{(n)} < 0$, and $y^{(n)}$ flips to -1. Now, the same procedure happens again, but with each variable increasing.
This gives oscillation. The problem comes when each oscillation takes longer than the previous one.
An example of this oscillation is depicted in Figure 4. Here, the system is unstable but still oscillating between -1 and +1 output. The system is unstable since the oscillations are exponentially increasing in both amplitude and period.

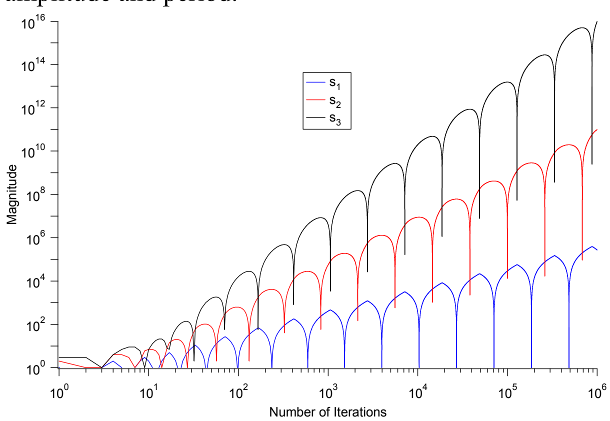

Figure 4. The absolute magnitude of the state space variables of an unstable third order sigma delta modulator with zero input. The results are presented on a log-log plot in order to show that the oscillations are exponentially increasing in period and amplitude.

6.3. Towards a linear programming approach for feedforward SDMs

Eq. (44) may be rewritten as

$$\mathbf{s}^{(n+1)} = \mathbf{A}\mathbf{s}^{(n)} + (u - y^{(n)})\mathbf{d} \qquad (48)$$

Thus, the state at any time may be given in terms of an initial state,

$$\mathbf{s}^{(n)} = \mathbf{A}^n\mathbf{s}^{(0)} + [\sum_{i=0}^{n-1}(u - y^{(i)})\mathbf{A}^{n-i-1}]\mathbf{d} \qquad (49)$$

So, for a given sequence, $y(0), y(1), ... y(N-1)$, we have the set of constraints

$$y^{(k)}\mathbf{c}^T\mathbf{s}^{(k)} \geq 0 \quad k = 0...n-1 \qquad (50)$$

In order to have a given output $y^{(k)}$,

$$y^{(k)}\mathbf{c}^T\mathbf{s}^{(k)} \geq 0 \qquad (51)$$

Substitution of (49) into (51) gives

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}^{(0)} + y^{(k)}\mathbf{c}^T[\sum_{i=0}^{k-1}(u - y^{(i)})\mathbf{A}^{k-i-1}]\mathbf{d} \geq 0 \qquad (52)$$

or

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}^{(0)} + y^{(k)}\mathbf{c}^T\sum_{i=0}^{k-1}\mathbf{A}^{k-i-1}\mathbf{d}u \geq y^{(k)}\mathbf{c}^T\sum_{i=0}^{k-1}y^{(i)}\mathbf{A}^{k-i-1}\mathbf{d} \qquad (53)$$

and the linear programming problem is to maximize u such that Eq. (52) holds. Linear programming problems are usually phrased such that all variables are restricted to positive values, so this can be accounted for by replacing **s** with $\mathbf{s}_1$- $\mathbf{s}_2$ where $\mathbf{s}_1$,$\mathbf{s}_2$ are positive.

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}_1^{(0)} - y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}_2^{(0)} + y^{(k)}\mathbf{c}^T\sum_{i=0}^{k-1}\mathbf{A}^{k-i-1}\mathbf{d}u \geq y^{(k)}\mathbf{c}^T\sum_{i=0}^{k-1}y^{(i)}\mathbf{A}^{k-i-1}\mathbf{d} \qquad (54)$$

This has been simulated using numerical recipes. Unfortunately, for typical high order SDMs, and any given bit sequence, the solution is unbounded. Further investigation is necessary.

7. CONCLUSION

The work described herein is concerned with the stability, or boundedness of SDMs. Ignoring more subtle questions such as the effect of dither, we have identified the following ten questions as fundamental issues in SDM stability theory.

1. For initial conditions set to zero, can one derive the value of constant input such that the magnitude of the state space variables will diverge towards infinity?
2. Given initial conditions and a constant input value, can one determine if this leads to bounded behavior?
3. Is there equivalence between bounded state variables and bounded bit sequences?
4. When is there initial condition dependence in the boundedness properties?
5. For a given SDM with given input, what are the bounds on the state variables?

6. If there is initial condition dependence for a given SDM with given initial conditions, what are the set of initial conditions which yield bounded behavior?
7. Can we characterize the stability (boundedness) properties of different SDM designs?
8. What is the relationship between the coefficients of an SDM and its boundedness?
9. What is the relationship between the order (dimensionality) of an SDM and its boundedness?
10. What can we say for all of the above when we don't assume constant input (we may assume input restricted to a certain range and/or bandlimited)?

Our approach has been to start with the simplest SDMs and work towards general results for more arbitrary and higher order designs. Though not fully described herein, this approach has been problematic since initial condition dependence is not significant in low order designs.

It is clear from Section 3 that the stability of first order SDMs may be easily derived. Second order SDMs are also tractable, based on existing literature and the derivations in this paper. However, some further results are necessary in order to add rigor to proofs of boundedness, derivation of the bounds, and applications to more general second order designs.

The analysis becomes more difficult when one considers third order designs, yet simulations indicate that simple relationships may exist between the SDM coefficients, the input and the boundedness of solutions. Furthermore, instability can be shown, using linear programming, for atleast one 3rd order SDM. Whether such techniques can be adapted more generally is as of yet unknown.

In general, unbounded solutions often fall into 2 categories. The less interesting of which exists for extremely high input $|u|>1$, where the state space variables diverge to infinity. More interesting are unstable solutions where $|u|<1$, and the state space variables continue to oscillate, but with exponentially increasing amplitude and period. This indicates a clear relationship between boundedness of the state space variables, and boundedness of the output bit sequences. Proving such an equivalence would go a long way towards improving the theory of SDM stability, and is a major goal of current research.

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