How Sigma Delta Modulators achieve high performance (and why they aren't even better)

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signal-to-noise ratio (SNR)

- single most used performance characteristic in A/D converters
- ratio of rms (root mean squared) signal to rms noise within the bandwidth of interest
  - $20\log_{10}$ of this ratio to derive SNR in decibels
  - Or $10\log_{10}$ of signal power over noise power
- SNR can be found for any A/D converter

- Approach
  - explain the basics of the signal-to-noise ratio (SNR),
  - theory and estimation in PCM and SDM converters
    - derive the formula for the SNR of an arbitrary SDM
  - Show simulated & theoretical SNR as a function of bits, order, OSR, input amplitude
    - Show how a high SNR is obtained
    - Show what prevents SDM implementations from achieving theoretical values

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Signal-to-noise ratio is probably the single most used performance characteristic in A/D converters. Here, we will explain the basics of the signal-to-noise ratio (SNR), its theory and estimation in PCM and SDM converters, and how it can be estimated in various situations.

The SNR is given by the ratio of rms (root mean squared) signal to rms noise within the bandwidth of interest. You then multiply the log10 of this ratio by 20 to derive SNR in decibels. The SNR can be found for any A/D converter. First, we derive the formula for the SNR of an ideal analog-to-digital converter.

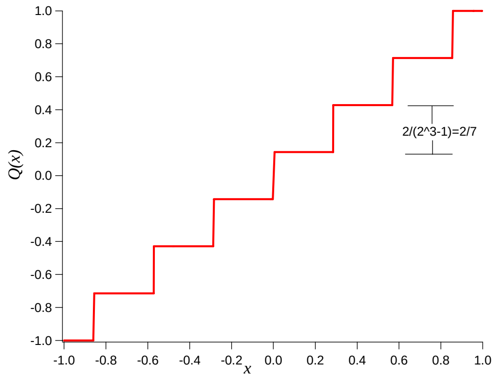
Quantization Assumption

- *quantization levels* - allowed values in output signal after quantization
- *quantization step size q*- distance between 2 successive levels
- quantiser with *b* bits covering range from +*V* to –*V*
  - 2^b quantization levels
  - width of each quantization step is

    $q=2V/(2^b-1)$
- quantization error - difference between input & output to quantiser

  $e_q=Q(x)-x$
- rounding quantizer - assigns each input to nearest quantization level

  $-q/2 \le e_q(n) \le q/2$



Transfer curve for b=3, V=1

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The allowed values in the output signal, after quantization, are called *quantization levels*, whereas the distance between 2 successive levels, *q*, is called the *quantization step size*. Consider a quantiser with *b* bits covering the range 2*V*, from +*V* to -*V*. Then there are 2^b quantization levels, and the width of each quantization step is

$$q=2V/(2^b-1)$$

This is depicted for a 3 bit quantiser and V=1.

The rounding quantizer assigns each input sample *x*(*n*) to the nearest quantization level. The quantization error is simply the difference between the input and output to the quantiser, $e_q=Q(x)-x$. It can easily be seen that the quantization error *eq*(*n*) is always bounded by +/- q/2

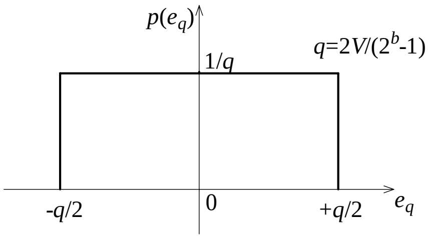
Quantization Assumption (2)

$$-q/2 \le e_q(n) \le q/2$$

- Assume quantisation error uniformly distributed over a quantisation step

$$p(e_q) = \begin{cases} 1/q & |e_q| \le q/2 \\ 0 & |e_q| > q/2 \end{cases}$$

- not exact
  - still quite close to accurate for noisy and some sinusoidal signals



Probability distribution function for the quantization error.

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We assume that the quantisation error is uniformly distributed over a quantisation step. This assumption is not exact, but is still quite close to accurate for sinusoidal signals. The pdf, probability distribution function for the quantization error, is depicted.

SNR for a PCM A/D Converter

- If sampling rate satisfies sampling theorem, $f_s > 2f_B$, quantization is only error in A/D conversion process
  - jitter and other effects are not considered here
- Assuming uniform distribution, average quantization noise is given by

$$\bar{e}_q = E\{e_q\} = \int_{-\infty}^{\infty} e_q\, p(e_q)\, de_q = \int_{-q/2}^{q/2} e_q \frac{1}{q}\, de_q = 0$$

- quantization noise power given by

$$\sigma_e^2 = E\{(e_q - \bar{e}_q)^2\} = E\{{e_q}^2\} = \int_{-\infty}^{\infty} {e_q}^2\, p(e_q)\, de_q = \int_{-q/2}^{q/2} {e_q}^2 \frac{1}{q}\, de_q = \frac{q^2}{12}$$

- Or

$$\sigma_e^2 = \frac{q^2}{12} = \frac{(2V/(2^b - 1))^2}{12} = \frac{V^2}{3(2^b - 1)^2} \approx \frac{V^2}{3 \cdot 2^{2b}}$$

- estimate signal power
  - assume we quantize sinusoidal signal of amplitude A, $x(t) = A\cos(2\pi t/T)$

$$\sigma_x^2 = E\{(e_x - \bar{e}_x)^2\} = E\{{e_x}^2\} = \frac{1}{T}\int_0^T (A\cos(2\pi t/T))^2\, dt = \frac{A^2}{2}$$

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If the sampling rate satisfies the sampling theorem, $f_s > 2f_B$, quantization is the only error in the A/D conversion process (jitter and other effects are not considered here). Using the assumption of uniform distribution, we can find the average quantization noise and the quantization noise power.

Quantisation noise power is found using textbook definitions

To find the SNR, we also need to estimate the signal power. Now assume we are quantizing a sinusoidal signal of amplitude A, $x(t) = A\cos(2\text{p}t/T)$. The average power of the signal is thus $A^2/2$

From and , the signal-to-noise ratio may now be given by,

Thus the signal-to-noise ratio increases by approximately 6dB for every bit in the quantiser. Using this formula, an audio signal encoded onto CD (a 16 bit format) using PCM, has a maximum SNR of 98.08. Also note from , that the SNR is linearly related to the signal strength in decibels.

SNR for a PCM A/D Converter (2)

- Quantization noise power $\sigma_e^2 = \frac{V}{3(2^b - 1)^2} \approx \frac{V}{3 \cdot 2^{2b}}$
- Signal power $\sigma_x^2 = A^2 / 2$
- signal-to-noise ratio,

$$\text{SNR(dB)} = 20\log_{10}\frac{\sigma_x}{\sigma_e} \approx 20\log_{10}\frac{A}{V} + 6.02b + 1.76$$

  - increases by 6dB for every bit in quantiser
  - linearly related to signal strength in decibels
- Audio signal encoded onto CD (16 bit format) using PCM
  - maximum SNR of 98.08.

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The signal-to-noise ratio may now be given.

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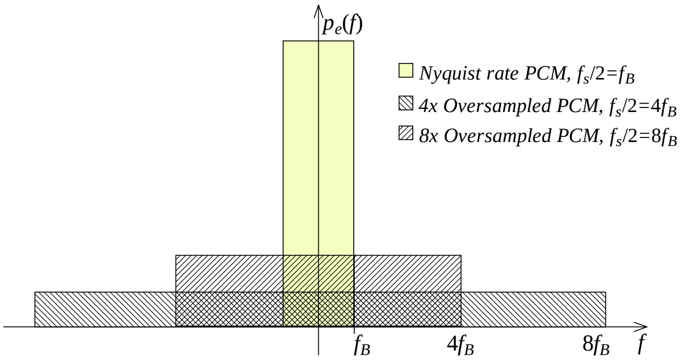
Oversampling

- Previously assumed
  - highest input frequency was $f_{B,}$ signal acquired at Nyquist rate, $f_s=2f_B$.
- now assume signal oversampled, $f_s=2^{r+1}f_B$
- Oversampling - sampling input signal at frequency much greater than nyquist
  - *decreases quantization noise in band of interest*



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SNR for an Oversampled PCM A/D Converter

- oversampling ratio is $OSR=2^r=f_s/2f_B$.
  - quantization noise spread over larger frequency range
  - only concerned with noise below Nyquist frequency
- *in-band* quantization noise power given by

$$\sigma_n^2 = \int_{-\infty}^{\infty} e_q^{\,2} p(e_q) de_q = \int_{-q/2}^{q/2} e_q^{\,2} \frac{2f_B}{f_s} \frac{1}{q} de_q = \frac{q^2}{2^r \cdot 12} = \sigma_e^2 / OSR$$

- Most noise power is now located outside of signal band
- signal power occurs over signal band only
  - remains unchanged
- SNR now given by

$$\mathrm{SNR}=10\log_{10}(\sigma_x^2/\sigma_n^2)=10\log_{10}(\sigma_x^2/\sigma_e^2)+10\log_{10}2^r$$
$$\approx 20\log_{10}(A/V)+6.02b+3.01r+1.76$$

- For every doubling of oversampling ratio, SNR improves by 3dB
  - 6dB improvement with each bit in quantiser remains
  - doubling oversampling ratio increases effective number of bits by ½ a bit

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The above discussion assumed that the highest possible frequency in the input signal was some value *fB* and that the signal is acquired at the Nyquist rate, *fs*=2*fB*. However, lets now assume that the signal is oversampled such that the rate is $f_s=2^{r+1}f_B$, that is, the oversampling ratio is OSR=2*r*=*fs*/2*fB*. Thus, the quantization noise is spread over a larger frequency range yet we are only concerned with noise below the Nyquist frequency.

The *in-band* quantization noise power is now given by the total quantisation noise power divided by the OSR.

Most of the noise power is now located outside of the signal band. The signal power occurs over the signal band only, so it remains unchanged.

The signal-to-noise ratio may now be given by, the previous SNR $+10\log_{10}$OSR

Thus for every doubling of the oversampling ratio, the SNR improves by 3dB. The 6dB improvement with each bit in the quantiser remains, so we can say that doubling the oversampling ratio increases the effective number of bits by ½ a bit.

SNR for an Oversampled PCM A/D Converter

- In-band quantization noise power can also be given by

$$\sigma_n^2 = \int_{-f_B}^{f_B} S_e(f)\,|NTF(f)|^2\,df = \frac{\sigma_e^2}{f_s}\int_{-f_B}^{f_B} |NTF(f)|^2\,df$$

- where power spectral density of (unshaped) quantization noise is

$$S_e(f) = \sigma_e^2 / f_s$$

- no noise shaping, just oversampling
  - noise transfer function is uniformly one over range [-f_B,f_B]

$$\sigma_n^2 = \frac{\sigma_e^2}{f_s}\int_{-f_B}^{f_B} 1df = \frac{2\sigma_e^2 f_B}{f_s} = \sigma_e^2 / OSR$$

- Same as before.
- Equivalent of 16 bit Nyquist rate PCM (98dB SNR)
  - 8bit – 2.64GHz

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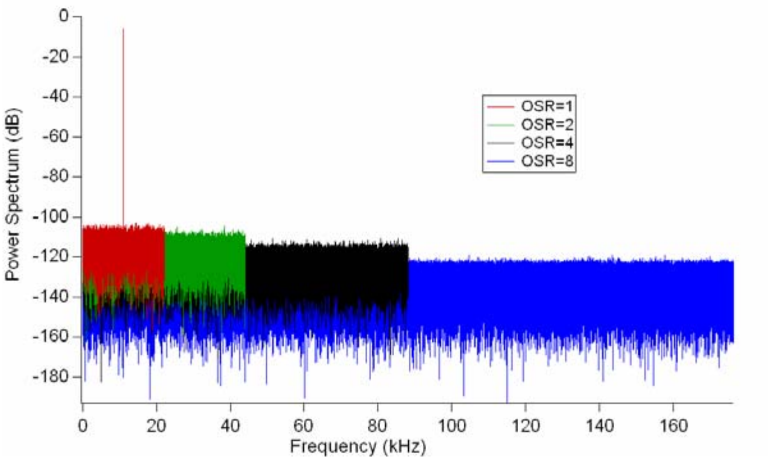
As a lead-in to the following sections which deal with sigma delta modulation, consider that the (in-band) quantization noise power can also be given as a function of the Noise Transfer Function and the power spectral density of the (unshaped) quantization noise. Since there is no noise shaping, just oversampling, we have that the noise transfer function is uniformly one over the range [-*fB*,*fB*]. So we get the same as .before

Oversampling

Doubling the oversampling ratio results in approximately a 3dB drop in the noise floor



12 bit PCM, full scale input at 11.025kHz, triangular PDF dither

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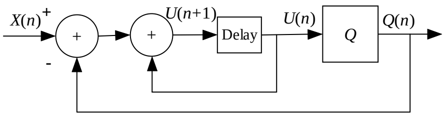
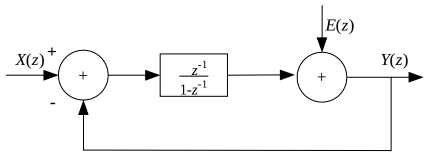
SNR for a 1st Order Sigma Delta Modulator

- pulse code modulators perform no noise shaping whatsoever
- first order sigma delta modulator.



Time-domain block diagram



z-transform block diagram, quantiser approximated by noise source

- given as $$U(n+1) = X(n) - Q(n) + U(n)$$

- Use linear model of SDM
  - Quantiser modeled as noise source, $e{=}Q{-}U$
- Time domain equation now given as

$$Q(n) = X(n-1) + Q(n) - U(n) - Q(n-1) + U(n-1)$$
$$= X(n-1) + e(n) - e(n-1)$$

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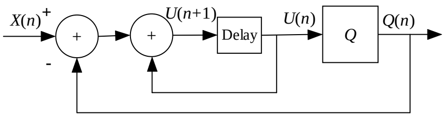
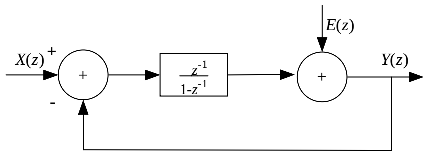
SNR for a 1st Order Sigma Delta Modulator

- first order sigma delta modulator



Time-domain block diagram

$$Q(n) = X(n-1) + e(n) - e(n-1)$$



z-transform block diagram, quantiser approximated by noise source

$$Y(z) = X(z)z^{-1} + E(z)(1-z^{-1})$$

- signal transfer function → z^{-1}
- noise transfer function → $1\text{-}z^{-1}$
  - $NTF(f) = 1 - e^{-j2\pi f/f_s}$
- So, $|NTF(f)|^2 = \ldots = 4\sin^2(\pi f/f_s)$

$$\sigma_n^2 = \int_{-f_B}^{f_B} S_e(f)\,|NTF(f)|^2\, df = \frac{\sigma_e^2}{f_s}\int_{-f_B}^{f_B} 4\sin^2(\pi f/f_s)df$$

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Here, $S_e(f)$ is the power spectral density of the unshaped quantization noise

SNR for a 1st Order Sigma Delta Modulator

- In-band, shaped quantization noise power

$$\sigma_n^2 = \sigma_e^2 \left[2/OSR - \frac{2}{\pi}\sin(\pi/OSR) \right] \approx \frac{\pi^2}{3 \cdot \mathrm{OSR}^3}\sigma_e^2$$

- SNR(dB)

$$10\log_{10}\frac{\sigma_x^2}{\sigma_n^2} \approx 20\log_{10}\frac{A}{V} + 6.02b + 9.03r - 3.41$$

- 9dB for each doubling of the oversampling ratio
  - Compare with 3dB improvement which occurs without noiseshaping
- Equivalent of 16 bit Nyquist rate PCM (98dB SNR)
  - 1bit – 96.78MHz

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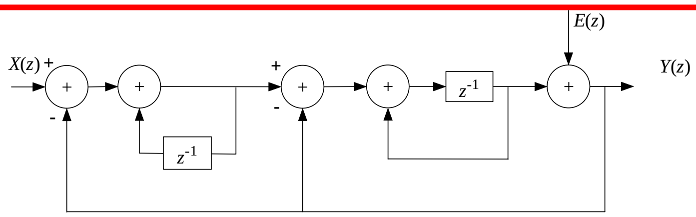
The effect of first order noise shaping is evident. We now get an improvement of 9dB for each doubling of the oversampling ratio, rather than the 3dB improvement which occurs without noiseshaping.

SNR for a 2nd Order Sigma Delta Modulator



- Transfer function

$$Y(z) = X(z)z^{-1} + E(z)(1 - z^{-1})^2 = STF(z)X(z) + NTF(z)E(z)$$

- Noise Transfer function

$$NTF(f) = (1 - e^{-j2\pi f/f_s})^2 = [2\sin(\pi f/f_s)]^2$$

- In-band, shaped quantization noise power

$$\sigma_n^2 = \frac{\sigma_e^2}{f_s}\int_{-f_B}^{f_B}[2\sin(\pi f/f_s)]^4 df \approx \frac{\pi^4}{5 \cdot OSR^5}\sigma_e^2$$

SNR for a 2nd Order Sigma Delta Modulator

$$\sigma_n^2 \approx \frac{\pi^4}{5 \cdot OSR^5} \sigma_e^2$$

- SNR (dB)

$$10\log_{10}\frac{\sigma_x^2}{\sigma_n^2} \approx 20\log_{10}\frac{A}{V} + 6.02b + 15.05r - 11.14$$

- 15dB improvement in the SNR with each doubling of the oversampling ratio
- Equivalent of 16 bit Nyquist rate PCM (98dB SNR)
  - 1bit – 6.12MHz

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Compared with the 1st order SDM, this provides more suppression of the quantization noise over the low frequencies, and more amplification of the noise outside the signal band.

Thus we see a large improvement in moving to a second order SDM. There is now a 15dB improvement in the SNR with each doubling of the oversampling ratio.

SNR for an N^{th} Order Sigma Delta Modulator

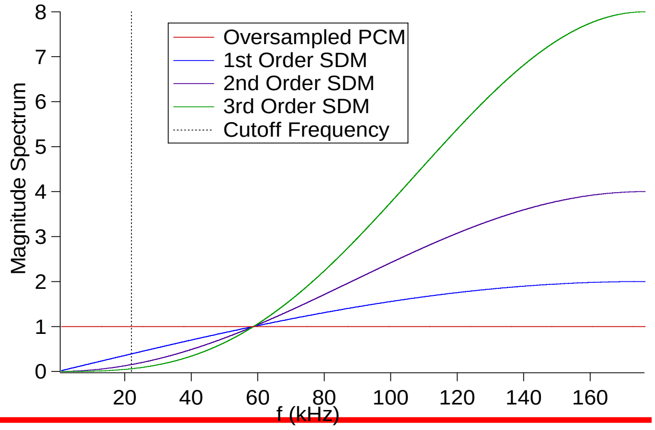
- Transfer function of a generic *N*th order SDM, is given by

$$Y(z) = X(z)z^{-1} + E(z)(1 - z^{-1})^N = STF(z)X(z) + NTF(z)E(z)$$

- Noise power in the baseband is given by

$$NTF(f) = (1 - e^{-j2\pi f / f_s})^N = [2\sin(\pi f / f_s)]^N$$

Noise Transfer functions for 8 times OSR



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The transfer function of a generic *N*th order SDM, is given by…

The noise power in the baseband is given by… as plotted.

SNR for an N^{th} Order Sigma Delta Modulator

$$NTF(f) = (1 - e^{-j2\pi f/f_s})^N = [2\sin(\pi f/f_s)]^N$$

- Using an integral formula, this gives

$$\sigma_n^2 = \frac{\sigma_e^2}{f_s}\int_{-f_B}^{f_B}[2\sin(\pi f/f_s)]^N\,df \approx \sigma_e^2\frac{\pi^{2N}}{(2N+1)OSR^{2N+1}}$$

- Giving $\text{SNR(dB)} = 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2} + 10\log_{10}\frac{(2N+1)2^{(2N+1)r}}{\pi^{2N}}$

$$\approx 20\log_{10}\frac{A}{V} + 6.02b + 1.76 + 10\log_{10}(2N+1) - 9.94N + 3.01(2N+1)r$$

- In general, for an *N*th order SDM
  - 3(2*N*+1)dB improvement in the SNR with each doubling of the oversampling ratio
  - 6dB improvement with each additional bit in the quantiser
  - → use of high order SDMs & high oversampling ratio offers much better SNR than simply increasing # bits

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The transfer function of a generic *N*th order SDM, is given by…

The noise power in the baseband is given by…

Using an integral formula, this gives…Giving

Thus in general, for an *N*th order SDM, there is a 3(2*N*+1)dB improvement in the SNR with each doubling of the oversampling ratio, and a 3dB improvement with each additional bit in the quantiser. Thus, use of high order SDMs and a high oversampling ratio offers a much better SNR than simply increasing the number of bits.

Of course, this is an approximation. It depends on the coefficients of the modulator, on the approximations used in the derivation, and other factors. Nevertheless, It provides an upper limit on the SNR, and many sigma delta modulators perform quite close to this limit.

For a 64 times oversampled 1-bit A/D converter, using a fifth order SDM, such as is typical in audio applications, we find that

I don't believe that anyone has ever designed a 5th order, 1 bit SDM which gives such a high SNR for a 64 times OSR. This is mainly because a design with near ideal noise shaping characteristics would be too unstable to allow any realistic input. But they have achieved SNRs of 120dB.

SNR for an N^{th} Order Sigma Delta Modulator

- this is an approximation
- depends on
  - coefficients of the modulator
  - on approximations used in derivation
    - Assume high oversampling ratio
    - High number of bits in quantiser
  - other factors.
- provides rough upper limit on SNR
- Some low order sigma delta modulators perform quite close to this limit

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SNR vs Bits

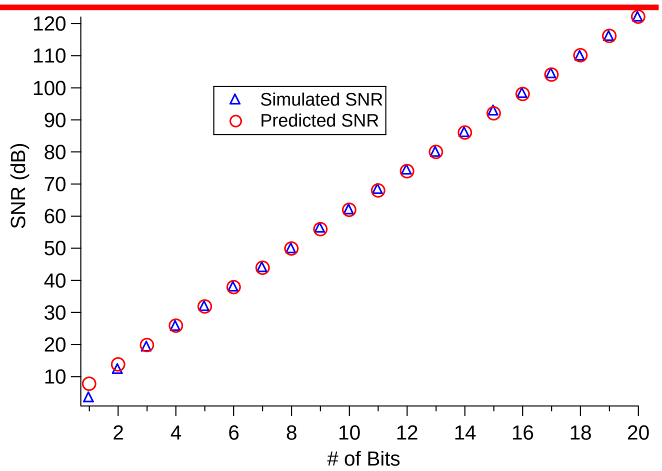
Predicted from

$$\text{SNR(dB)} = 20\log_{10}\frac{\sigma_x}{\sigma_e}$$

$$\approx 20\log_{10}\frac{A}{V} + 6.02b + 1.76$$

assumes

$$\sigma_e^2 = \frac{V^2}{3(2^b - 1)^2} \approx \frac{V^2}{3 \cdot 2^{2b}}$$



SNR as function of # bits in quantiser for PCM encoded signal, sampled at Nyquist frequency

input freq. 2kHz , full range -1 to 1, sampling rate 44.1kHz

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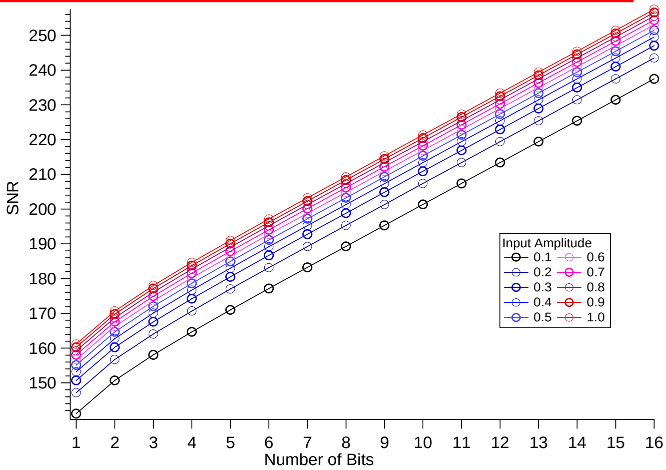
Effect of bits and signal amplitude

- 5th Order, 64 times OSR
- From the formula using

$$\sigma_e^2 = \frac{V^2}{3(2^b - 1)^2}$$



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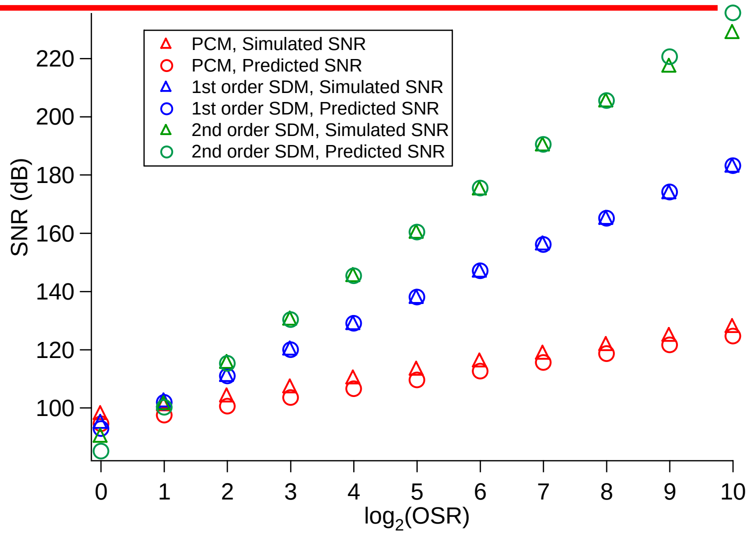
SNR vs Oversampling Ratio

Assumptions
1. high OSR→
lack of agreement for 1, 2, and 4 times oversampling.

2. Precision issues for high SNR →
disagreement between theoretical & simulated results for 2nd order SDM at OSR=512 or 1024

3. uniform noise distribution→
constant error between theory & simulation for PCM

3, 9 & 15dB increases for doubling OSR confirmed



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Why aren't SDMs close to the ideal?

- Example
  - 64 times oversampled 1-bit A/D converter, using 5th order SDM

$$\mathrm{SNR(dB)} \approx 20\log_{10}\frac{A}{V} + 167.15$$

  - No 5th order, 1 bit SDM gives such high SNR for 64 times OSR
- Many assumptions, but almost all lead to only minor differences
  - #bits>>0
  - OSR>>0
  - Uniform quantisation noise distribution
- Design with near ideal noise shaping characteristics would be too unstable to allow any realistic input
  - But they have achieved SNRs of 120dB.

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