From Time Series to Symbolic Dynamics: An Algebraic Topological Approach

K. Mischaikow
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30030 USA
mischaik@math.gatech.edu

M. Mrozek*
Instytut Informatiyki
Uniwersytet Jagielloński
30-072 Kraków, Poland
mrozek@ii.uj.edu.pl

A. Szymczak
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30030 USA
andrzej@math.gatech.edu

J. Reiss
School of Physics
Georgia Institute of Technology
Atlanta, GA 30030 USA

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Abstract

A new approach to constructing a dynamical systems model from experimental time series data is presented. Using the the ideas of delay reconstruction a multivalued dynamical system is constructed. The multivalued approach is taken to allow for bounded experimental error. This system is then analyzed and algebraic invariants based on the Conley index are computed. These invariants have implications concerning the dynamics which must occur, e.g. symbolic dynamics and horseshoes. Theorems indicating conditions under which these invariants can be lifted back to the original physical system are then proven.

1 Introduction

The goal of this paper is to introduce some new ideas into the problem of detecting and modeling chaotic dynamics in physical systems. To focus attention on the difficulties we begin with a description of a well known device used by physicists to test nonlinear dynamic concepts. The experiment consists of a vertically oriented magnetoelastic ribbon (the dimensions of the material resemble that of Christmas tinsel) placed within a vertical time periodic magnetic field. Without the magnetic field the weight of the ribbon causes it to buckle. However, under a weak magnetic field, the stiffness of the ribbon changes by an order of magnitude, causing it to straighten. The data from this experiment is taken at the same frequency as the forcing of the magnetic field and consists of the output voltage of an optical censor which measures the displacement of the ribbon near its base.

A very natural procedure to model this system is as follows. Since we are interested in the macroscopic behavior of the ribbon, the theory of rational mechanics can be used to obtain a time dependent periodically forced fourth order nonlinear partial differential equation. Ideally, as mathematicians we would provide a rigorous description of the dynamics of this infinite dimensional system. Unfortunately, this appears to be beyond the current capabilities of the mathematics community. Therefore, to continue we need to simplify our model further. The simplest (and most common) method is to perform a Galerkin approximation to the partial differential equation;

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the result being a periodically forced ordinary differential equation. Again, this system is probably still beyond rigorous analysis and hence one is led to numerical computations to understand the dynamics. [1]

Observe that we have introduced at least three sources of error through this form of analysis. The first is the neglection of microscopic properties of material to obtain the partial differential equation. The second is in the Galerkin approximation and the third is in the numerical method used to simulate the dynamics. There is yet another source of error which will have to be dealt with and that is the experimental errors.

At this point we have numerical data obtained from our model and experimental data provided by the physicist. The classical scientific paradigm requires us to compare the predictions of the model with the results of the experiment. However, our expectation is that this process exhibits chaotic dynamics. More precisely, an arbitrarily small change in the initial condition will lead to significant changes in the output; this applies both to the experimental data and, assuming that we have a reasonably accurate model, to the numerics. Therefore, pointwise agreement between the numerical and experimental data would strongly indicate that the process was not chaotic. Thus we are in the somewhat paradoxical situation that significant pointwise disagreement between the numerics and the data is a necessary condition for the adequacy of the model. The question which remains is what, within the classical paradigm is a sufficient condition?

We suggest a complementary approach; use the experimental time series data to directly generate a model (in the form of symbolic dynamics) for the system. The focus of this paper is on clearly describing the assumptions being made on the relationships between the physical system, the experimental data, and the model which is constructed. Therefore, in Section 2 we shall begin presenting a series of formal definitions and assumptions addressing these issues. Obviously, in the context of physical experiments it is impossible to rigorously verify the validity of the assumptions. Never the less we adopt

[1] An analysis of this type was performed by Moon and P. Holmes for a different version of this experiment [16, 6]. In their case they provide a convincing argument that a Duffing equation of the form

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos \omega t$$

provides the simplest possible model for their experiment. It is obtained by a single mode Galerkin approximation of an elastic beam equation. Of course their numerical and experimental data do not agree pointwise.

this formal mode of presentation in an attempt to clarify these principles. The hope is that this clarification can aide the experimentalist in deciding to what extent the model generated by our methods can be trusted and also to what extent *any* analysis of the data can be trusted. Furthermore, as shall be indicated the assumptions are fairly weak and denying their validity may raise other concerns about the interpretation of the data.

Before plunging into the details, we shall give a heuristic description of our approach. We think of the physical system as a *flow*

$$\varphi : \mathbf{R} \times X \to X$$

on a topological space X, i.e. φ is a continuous map with the properties that $\varphi(0, x) = x$ and $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$. We are interested in the existence and structure of *invariant sets*; that is subsets $S \subset X$ with the property that

$$\varphi(\mathbf{R}, S) = S.$$

An essential difficulty is that while the invariant set of interest S may be a fairly simple set, e.g. a periodic orbit, or a complicated but low dimensional set, e.g. a strange attractor, the ambient space X may be a very large or high dimensional space. This has led to a variety of techniques that allow one to study the dynamics restricted to S.

Conceptually inertial manifolds provide one of the simplest approaches. Typically one considers a partial differential equation in which case X is a Banach space. The first step is to find an appropriate decomposition of X into linear subspaces, i.e. $X = Z \oplus Z^c$ where Z is finite dimensional, and hence, homeomorphic to $\mathbf{R}^n$ for some n. An inertial manifold S is a normally hyperbolic invariant set which can be expressed as the graph of a smooth function $h : Z \to Z^c$, i.e. $S =$graph(h). Observe that the projection of S onto Z can be thought of as mapping the dynamics on $S \subset X$ bijectively onto $\mathbf{R}^n$. Using this procedure the partial differential equation restricted to S is reduced to an ordinary differential equation. This is similar in spirit to the Galerkin approximation, with the important difference that it is an exact representation of the dynamics on S in X.

On a slightly more general level, if the invariant set $S \subset X$ is an arbitrary manifold then one can attempt to find an embedding $h : S \to \mathbf{R}^n$. A fundamental result of differential topology is that if S is compact, smooth, has dimension k and $n \geq 2k+1$, then the "typical" map h is an embedding.

With this in mind, given a dynamical system φ, an invariant set S, and a collection of points lying on a trajectory in S the question becomes, how does one construct such a map h? To answer this we begin by assuming that the data was collected at regular time intervals, say at every $\tau > 0$ unit of time. Observe that this is equivalent to studying the discrete time dynamical system

$$\begin{aligned} f : X &\rightarrow X \\ x &\mapsto f(x) := \varphi(\tau, x) \end{aligned}$$

Within this framework Ruelle advocated the use of delay reconstruction coordinates. This was implemented first by Packard *et al.* [19] and the procedure is as follows. Starting with a time series (u_i), $i = 1, \ldots, N$, that is a collection of scalar values obtain by measurements at regular time intervals, one chooses an embedding dimension n and then constructs vectors

$$y_i = (u_i, u_{i+1}, \ldots, u_{i+n-1}) \in \mathbf{R}^n.$$

This approach was first put on a mathematical footing by Takens [26] and then significantly generalized by Sauer, Yorke, and Casdagli [22]. Under the assumption that τ is fixed and that there is no noise in the experimental data they showed that if n is chosen sufficiently large then this procedure will typically embed S, an invariant set for the physical system, in $\mathbf{R}^n$. It should be noted that this approach has become a cornerstone of analysis of nonlinear experimental data [1].

There are of course two essential assumptions underlying the above mentioned results: a fixed sampling rate and no error in the time series data. In practice these assumptions can never be satisfied. Our goal, therefore, is to expand upon these ideas and apply invariants of dynamical systems which are robust with respect to moderate perturbations. Our approach is based on three essential concepts.

The first is that of an *isolating neighborhood*. This is a compact set $N \subset X$ such that $\mathrm{Inv}(N)$, the maximal invariant set in N, is contained in the interior of N. This is equivalent to the statement that given any point $x \in \partial N$ there is some finite time, either positive or negative, under which the image or preimage of x at that time is *not* contained in N. Given the compactness of N it is now a simple continuity argument to show that if N is an isolating neighborhood for a given dynamical system, then N

is an isolating neighborhood for all sufficiently nearby dynamical systems. Thus isolating neighborhoods are robust with respect to sufficiently small perturbations (think experimental errors).

A second idea is needed to translate information about isolating neighborhoods back to the dynamics of invariant sets. For this we use the Conley index theory. This is an index for isolating neighborhoods and as such is also stable under perturbation. As will be discussed briefly in Section 4, this index is, for our purposes, an algebraic topological quantity. Furthermore, recent abstract results [3, 4, 10, 23, 24] demonstrate that the algebra can be used to indicate the existence of chaotic dynamics in the sense of symbolic dynamics. More closely related to the ideas of this paper are the applications of this theory to computer assisted proofs of the existence of chaotic dynamics in the Lorenz equations [11, 12, 15] and the Henon map [25]. In these settings the perturbations are the numerical errors which can be bounded, but obviously are not known precisely.

Finally, while as was indicated above we have mathematical objects which are robust with respect to perturbations, we need some way of determining a priori how large a perturbation is admissible. To overcome this we use multivalued dynamical systems. In particular, there is a Conley index theory for multivalued dynamics with the important property that an index computed for the multivalued system is also valid for any dynamical system arising from a continuous selector of the multivalued system.

In summary then our approach is to use the delay reconstruction to construct a multivalued dynamical system in $\mathbf{R}^n$. This multivalued system is meant to represent all dynamics close to the original physical system. The Conley index is then computed for the multivalued system. This of course implies that the Conley index has been computed for any continuous selector of the multivalued system. At this point there are two possible interpretations of our results. The first is to assume that an embedding of the invariant set of the physical system is such a continuous selector, in which case one can directly apply the index results to conclude the existence of symbolic dynamics. Alternatively, one may try to lift the algebra of the index back to the original phase space X and apply the index theory results to the dynamics on X. The advantage of this is that this can be done under weaker assumptions than that of an embedding. The disadvantage is that the conclusions are in general weaker.

As is indicated in the title and earlier in this Introduction, we are interested in obtaining a description of chaotic dynamics in terms of symbolic dynamics. In practice this is done by finding an isolating neighborhood for the multivalued dynamical system in $\mathbf{R}^n$ which is the finite union of disjoint compact subsets. Each of these subsets is then assigned a symbol and the Conley index information is used to determine the associated symbolic dynamics for the multivalued system. Finally, this symbolic dynamics is lifted back to the physical system acting on X in the following sense. Under the assumptions which will be presented, one may conclude that for every admissible symbol sequence in the multivalued system, there exists a trajectory in the phase space X such that it measurements would generate the corresponding symbolic dynamics in $\mathbf{R}^n$.

The outline of the paper is as follows. As was indicated above Section 2 presents the assumptions on the physical system and the measurements. Section 3 describes a method for generating the multivalued map. Section 4 provides a brief discussion of the Conley index, a general theorem describing when the index can be lifted, and the implication for the observable dynamics. In Section 5 another simpler approach to lifting the dynamics back to X is presented.

One final comment is that this method has been used successfully to analyze data from the experiment described in the first paragraph of the Introduction. The interested reader is referred to [13, 14].

2 Experimental Assumptions

We shall present in this section a series of assumptions concerning the physical system and the experimental measurements plus some comments concerning these assumptions. As was remarked in the introduction in an experimental setting we cannot hope to verify these assumptions, but present them as a check to the experimentalist.

Assumption A1. *The physical process can be modeled by a continuous map*

$$\begin{aligned} f : X \times \Lambda &\rightarrow X \\ (x, \lambda) &\mapsto f(x, \lambda) = f_\lambda(x) \end{aligned}$$

where X and Λ are topological spaces.

The topological space X should be thought of as the phase space for the system. Λ represents the parameter space for the experiment. Obviously, since f is continuous in both X and Λ, fixing $\lambda \in \Lambda$ produces a continuous dynamical system

$$f_\lambda : X \rightarrow X.$$

Observe that at this point given any fixed parameter value λ, we have made a fairly strong assumption on the dynamics of f_λ. In particular, it is not stochastic. Our justification for this is that if one is interested in macroscopic phenomenon, such as the behavior of the magnetoelastic ribbon described in the introduction, and furthermore, if one is only interested in moderate time length descriptions of such phenomena, then one expects that a deterministic model should suffice.

Definition 2.1 An *experiment* is a collection of points in X and parameter values in $\Lambda_E \subset \Lambda$,

$$E := \{(x_j, \lambda_j) \in X \times \Lambda_E \mid 0 \leq j \leq J\},$$

satisfying $x_{j+1} = f(x_j, \lambda_j)$.

An important observation at this point is that we allow for the parameter values λ_j to vary randomly. This appears to be the only physically realistic assumption that is possible. The experimentalist can never have complete control on the experiment or exactly choose parameter values. Therefore, the only constraint that we impose on the parameter values is that during the experiment they can be controlled to lie within $\Lambda_E \subset \Lambda$. An implication of this is that if we are given a point $x \in X$ then after one iteration all we can assume is that it lies in

$$f_{\Lambda_E}(x) := f(\{x\} \times \Lambda_E) \subset X.$$

A second reason for insisting that parameter values be allowed to vary randomly over some Λ_E is that physically it is impossible to sample at precise time intervals. Therefore, if one views the sampling as choosing points along a trajectory of φ_τ, then small uncontrolled changes in the time of sampling can be thought of as being equivalent to choosing points along a trajectory obtained by the sequence of maps φ_{τ_0}, φ_{τ_1}, φ_{τ_2}, etc. By continuity with respect to time these maps are small perturbations from φ_τ and therefore can be incorporated into Λ_E assuming the space of perturbations Λ is chosen large enough.

As was indicated in the introduction, the process of measuring an experiment inherently involves errors. Therefore, we model the measurements as follows.

Definition 2.2 A *measurement* is a multivalued map

$$\begin{array}{rcl} \theta : X & \to & \text{intervals in } \mathbf{R} \\ x & \mapsto & [a_x, b_x] \subset \mathbf{R}. \end{array}$$

Observe that the measurement only depends upon the point in phase space. Therefore, up to the error tolerance of the measurement, $b_x - a_x$, the measurements are time independent.

In order for the experimental data to have any meaning there must be some amount of continuity in the measurement map. If not, then arbitrarily small changes in the physical system could result in instantaneous arbitrarily large changes in measurements. We choose to impose this restriction by assuming that a hypothetical "true" measurement, $\gamma(x) \in \mathbf{R}$ exists even if it cannot be performed. The precise statement of this is as follows.

Assumption A2. *There exists a measurement function θ and a continuous function*

$$\gamma : X \to \mathbf{R}$$

such that

$$\gamma(x) \in \theta(x) \quad \forall\ x \in X.$$

In the language of multivalued maps, γ is a continuous selector for θ.

Remark 2.3 We could at this point have assumed that several quantities could be measured simultaneously. In this case θ would be a multivalued map taking products of intervals as values and the true measurement would be a vector valued function. Aside from complicating the notation, this would have no effect on the results of this paper and hence will not be mentioned again.

We are finally in a position to describe what is meant by time series.

Definition 2.4 A *time series data set* is a collection of numbers

$$\Sigma = \{u_j^i \in \mathbf{R} \mid 1 \leq j \leq J^i,\ 1 \leq i \leq I\}$$

for which there exist experiments

$$E^i = \{(x_j^i, \lambda_j^i) \in X \times \Lambda_E \mid 1 \leq j \leq J^i\}, \quad 1 \leq i \leq I$$

such that $u_j^i \in \theta(x_j^i)$. The time series data is *sampled from* $A \subset X$ if, additionally, $x_j^i \in A$ for all $1 \leq j \leq J^i$ and $1 \leq i \leq I$.

Observe that the only relationship assumed between the time series data and the "true" physical process is that pointwise

$$|u_j^i - \gamma(x_j^i)| \leq b_{x_j^i} - a_{x_j^i}.$$

From a probabilistic point of view we are only allowing for bounded perturbations.

3 Modeling the Dynamics

We are now in the position of assuming that we have been given a time series data set Σ as defined in the previous section and want to construct a model of the dynamics. As was indicated in the introduction the strategy is to use a delay reconstruction to create a topological space on which a multivalued dynamical system will be defined and then to use this dynamical system to model the physics. Obviously with a finite set of experimental data points it is impossible to rigorously determine if we have captured the correct dynamics. Therefore, to explain the ideas which motivate our approach we begin in the idealized setting of assuming perfect knowledge. This is then used to determine what properties our multivalued map should posses.

It should be noted at this point that there are actually two possible interpretations to the dynamics of the physical system. The first is to fix a particular parameter value and consider the dynamics of f_λ. Using an unknowable true measurement, given an initial condition $x \in X$ the delay reconstruction map would take the form

$$\begin{aligned} \Gamma_\lambda : X &\to \mathbf{R}^n \\ x &\mapsto (\gamma(x), \gamma(f_\lambda(x)), \ldots, \gamma(f_\lambda^{n-1}(x))). \end{aligned}$$

The other (more general) interpretation is that the parameters change randomly in time. In this case one picks an arbitrary but fixed random sequence of parameter values $\{\lambda_j\} \subset \Lambda_E$ and $j_0 \in \mathbf{Z}$. Again, using an unknowable true measurement and given an initial condition $x \in X$ at the ${j_0}^{th}$ time step, the delay reconstruction map would take the form

$$\begin{aligned} \Gamma^{j_0}_{\{\lambda_j\}} : X &\to \mathbf{R}^n \\ x &\mapsto (\gamma(x), \gamma(f_{\lambda_{j_0}}(x)), \ldots, \gamma(f_{\lambda_{j_0+n-1}} \circ f_{\lambda_{j_0+n-2}} \circ \ldots \circ f_{\lambda_{j_0}}(x))). \end{aligned}$$

Observe that at this point we have already chosen and fixed n, the dimension of the reconstruction space dynamics. This is perhaps the most suspect step in the entire procedure in the sense that there are few generally reliable methods for determining the appropriate reconstruction dimension.

Also, observe that the reconstruction maps defined above are not what will be experimentally observed. To capture all possible experimental observations given assumptions **A1** and **A2** we define the multivalued map

$$M_n : X \rightrightarrows \mathbf{R}^n$$

$$x \mapsto \bigcup_{(x_1,x_2,\ldots,x_n)\in T_n(x)} \theta(x_1)\times\ldots\times\theta(x_n)$$

where

$$T_n(x) = \Big\{(x_1,x_2,\ldots,x_n)\in X^n \mid x_1 = x \text{ and there exist } \lambda_1,\ldots,\lambda_{n-1}\in\Lambda_E \text{ such that } x_{i+1} = f_{\lambda_i}(x_i) \text{ for } i = 1,2,\ldots,n-1\Big\}.$$

The range of M_n clearly contains all possible values that could be achieved through experimentation.

In practice the points which come from experiments usually are not spread out over the whole range of M_n. Instead they tend to accumulate around a much smaller region of $\mathbf{R}^n$. We shall assume that this kind of behavior is a result of the existence of an attracting set A ('attracting' means that $f_{\Lambda_E}(x)\subset A$ for any $x\in A$) of the physical system which is approached by the sequence of its states in progress of the experiment.

Clearly, f induces a multivalued dynamical system Ω_n on $O_n = M_n(A)$ defined by

$$\Omega_n(y) := \{v\in O_n \mid \exists x\in A \text{ such that } y\in M_n(x), \exists\lambda\in\Lambda_E \text{ such that } v\in M_n(f_\lambda(x))\}$$

Given only the constraints implicit in **A1** and **A2**, Ω_n represents the optimal knowledge of the dynamics on A as viewed through an n-dimensional delay reconstruction. This can be seen by realizing that given y a vector obtained from n consecutive data points, $y = (u_j, u_{j+1},\ldots u_{j+n-1})$, the best safe prediction that can be made concerning u_{j+n} is that $(u_{j+1},\ldots,u_{j+n})\in\Omega_n(y)$. Any set smaller than $\Omega_n(y)$ fails to contain a possible image point. Unfortunately, it is impossible in practice to compute Ω_n. Therefore, we weaken our expectations concerning the dynamics which will be studied on $\mathbf{R}^n$.

Definition 3.1 A multivalued map $F: D \overrightarrow{\to} D$ (where $D\subset\mathbf{R}^n$) is an *envelope* of f if, for some true measurement γ, the following two conditions are satisfied for any sequence of parameter values $\{\lambda_j\}\subset\Lambda_E$ and $j_0\in\mathbf{Z}$.

1. $\Gamma^{j_0}_{\{\lambda_j\}}(A)\subset D$,

2. The diagram

$$\begin{array}{ccc} A & \overset{f_{\lambda_{j_0-1}}}{\to} & A \\ \downarrow \Gamma^{j_0}_{\{\lambda_j\}} & & \downarrow \Gamma^{j_0}_{\{\lambda_j\}} \\ D & \overset{F}{\rightrightarrows} & D \end{array}$$

upper-semicommutes i.e., for every $x \in A$,

$$\Gamma^{j_0}_{\{\lambda_j\}} \circ f_{\lambda_{j_0-1}}(x) \in F \circ \Gamma^{j_0}_{\{\lambda_j\}}(x).$$

The following two propositions are straightforward.

Proposition 3.2 Ω_n *is an envelope of* f.

Proposition 3.3 *Let* $F : D \rightrightarrows D$ *have the property that* $\Omega_n(y) \cap D \subset F(y)$ *for all* $y \in D$. *Then,* F *is an envelope of* f*, provided there is a true measurement such that* $\Gamma^0_{\{\lambda_j\}}(A) \subset D$ *for any sequence* $\{\lambda_j\} \subset \Lambda_E$.

It it possible to state a sufficient condition for a multivalued map to be an envelope of f in terms of the *modulus of expansion of* Ω_n, i.e. the function $\epsilon : [0, \infty) \to [0, \infty)$ defined by

$$\epsilon(\delta) := \sup\Big\{\text{dist}(w_1, w_2) \mid \exists_{v_1,v_2} \in O_n \ \text{dist}(v_1, v_2) \leq \delta, \\ w_1 \in \Omega_n(v_1), w_2 \in \Omega_n(v_2)\Big\}.$$

Fix the reconstructed time series data

$$\Sigma_n := \{(u^i_j, u^i_{j+1}, \dots u^i_{j+n-1}) \in \mathbf{R}^n \mid 0 \leq j \leq J^i - n + 1, 1 \leq i \leq I\}.$$

Let

$$\mathring{\Sigma}_n := \{(u^i_j, u^i_{j+1}, \dots u^i_{j+n-1}) \in \mathbf{R}^n \mid 0 \leq j \leq J^i - n, 1 \leq i \leq I\}.$$

and $T : \mathring{\Sigma}_n \rightrightarrows \Sigma_n$ be the shift map defined by

$$T(y) = \Big\{(u^i_{j+1}, u^i_{j+2}, \dots u^i_{j+n}) \mid i, j \text{ are such that } y = (u^i_j, u^i_{j+1}, \dots u^i_{j+n-1})\Big\}.$$

We have the following.

Proposition 3.4 *Let $F : D \rightrightarrows D$, $D \subset \mathbf{R}^n$ be a multivalued map. If the first condition of Definition 3.1 is satisfied and:*

1. $D \subset \bigcup_{\eta \in \overset{\circ}{\Sigma}_n} B_\delta(\eta)$,
2. $\forall_{\eta \in \overset{\circ}{\Sigma}_n} \forall_{\eta' \in B_\delta(\eta)}\ D \cap B_{\epsilon(\delta)}(T(\eta)) \subset F(\eta')$,
3. *the time series is sampled from A,*

then F is an envelope of f.

Proof. Let $y \in D$. There exists $\eta \in \overset{\circ}{\Sigma}_n$ such that $\mathrm{dist}(y, \eta) \leq \delta$. By the definition of the modulus of expansion (notice that $T(\eta) \in \Omega_n(\eta)$ since the time series is sampled in A),

$$\Omega_n(y) \cap D \subset B_{\epsilon(\delta)}(T(\eta)) \cap D \subset F(y).$$

By Proposition 3.3, the proof is finished. □

Before proceeding further it is worth examining the hypotheses of the above proposition. The first is a requirement that the experimental data set approximate a sufficiently large portion of D. In principle one should be able to satisfy this condition by running the experiment for a sufficiently long time or sufficiently many times. Unfortunately, there is no *a prior* method for determining these times. The second hypothesis is also not rigorously verifiable in practice, but for reasonable physical systems one expects that using sufficiently many data points one can obtain a reasonable approximation to the expansion rates. In fact, approximations of this form are by now standard in the analysis of time series [1].

Our goal, therefore, is to generate a multivalued map $F : D \rightrightarrows D$ ($D \subset \mathbf{R}^n$) which is finitely representable, captures the information provided by the time series data and (under some additional assumptions) allows one to make interesting statements about the dynamics of the physical system on A.

The fundamental obstacle to reconstructing the dynamics of f is that we have no way of knowing the dynamics of points outside $\overset{\circ}{\Sigma}_n$. This will be dealt with later by making several assumptions.

For the moment, observe that since Σ_n is a discrete set of data points it does not posses an interesting topology and one must be imposed. This can be done in a variety of ways (see [9, 18]). The approach we take is perhaps the simplest and is based on grids in $\mathbf{R}^n$. Let $\alpha > 0$ be a fixed number. To each $k = (k_1, k_2, \ldots, k_n) \in \mathbf{Z}^n$ we identify a grid element

$$G_k = G_{(k_1,k_2,\ldots,k_n)} := \prod_{i=1}^{n} [k_i\alpha, (k_i+1)\alpha] \subset \mathbf{R}^n.$$

The set of all grid elements is denoted by $\mathcal{G}$. We are only interested in those grid elements which correspond to experimental data and so we define

$$\mathcal{G}_{\overset{\circ}{\Sigma}_n} := \{G_k \mid \overset{\circ}{\Sigma}_n \cap G_k \neq \emptyset\}.$$

The above set is indexed by

$$Z_{\overset{\circ}{\Sigma}_n} := \{k \in \mathbf{Z}^n \mid G_k \in \mathcal{G}_{\overset{\circ}{\Sigma}_n}\}.$$

Given $K \subset \mathbf{Z}^n$, let

$$|K| := \bigcup_{k \in K} G_k.$$

For any multivalued function $\mathcal{T} : Z_{\overset{\circ}{\Sigma}_n} \rightrightarrows Z_{\overset{\circ}{\Sigma}_n}$ we define the multivalued map $\lfloor \mathcal{T} \rfloor : |Z_{\overset{\circ}{\Sigma}_n}| \rightrightarrows |Z_{\overset{\circ}{\Sigma}_n}|$ by

$$\lfloor \mathcal{T} \rfloor(y) = |\bigcap \{\mathcal{T}(k) \mid k \in Z_{\overset{\circ}{\Sigma}_n} \text{ is such that } y \in G_k\}|.$$

We shall model the physical system with a multivalued map of the above form. The advantage of this approach is that such maps have obvious finite representations. Clearly, we would like to model the system in a way which does not contradict the information supplied by the time series data. This motivates the following definition.

Definition 3.5 A multivalued map $\mathcal{T} : Z_{\overset{\circ}{\Sigma}_n} \rightrightarrows Z_{\overset{\circ}{\Sigma}_n}$ is said to be *consistent with the time series data* if and only if the following condition is satisfied.

> If $k, k' \in Z_{\overset{\circ}{\Sigma}_n}$ are such that there exists $y \in \overset{\circ}{\Sigma}_n$ with $y \in G_k$ and $T(y) \cap G_{k'} \neq \emptyset$ then $k' \in \mathcal{T}(k)$.

Clearly, there are a lot of maps consistent with the time series data. As will be shown in the following sections, some of them (but not all) allow one to make interesting statements about the dynamics of the physical system. Roughly speaking, apart from being envelopes, such maps have to be large enough to carry homology, but, at the same time, small enough to allow one to prove isolation.

We have the following simple proposition whose proof is left to the reader.

Proposition 3.6 *Assume that* $\mathcal{T} : Z_{\overset{\circ}{\Sigma}_n} \rightrightarrows Z_{\overset{\circ}{\Sigma}_n}$ *is such that* $\lfloor \mathcal{T} \rfloor$ *is an envelope of* f*. Then,* $\mathcal{T}$ *is consistent with the time series data.*

We finish this section with a description of a procedure that we shall use to model the dynamics of f on $|Z_{\overset{\circ}{\Sigma}_n}|$. First of all, let us define the map $\mathcal{T}_0 : Z_{\overset{\circ}{\Sigma}_n} \rightrightarrows Z_{\overset{\circ}{\Sigma}_n}$ by

$$\mathcal{T}_0(k) := \left\{ k' \in Z_{\overset{\circ}{\Sigma}_n} \;\middle|\; \text{there is } y \in \overset{\circ}{\Sigma}_n \text{ with } y \in G_k \text{ and } T(y) \cap G_{k'} \neq \emptyset \right\}.$$

One can easily see that the above map is the smallest map consistent with the time series data. More precisely, we have the following easy proposition.

Proposition 3.7 *A multivalued map* $\mathcal{T} : Z_{\overset{\circ}{\Sigma}_n} \rightrightarrows Z_{\overset{\circ}{\Sigma}_n}$ *is consistent with the time series data if and only if* $\mathcal{T}_0(k) \subset \mathcal{T}(k)$ *for each* $k \in Z_{\overset{\circ}{\Sigma}_n}$*.*

For a finite set $\mathcal{B} \subset Z_{\overset{\circ}{\Sigma}_n}$ and $d \in \mathbf{Z}^+$ let

$$\begin{aligned} \mathrm{rct}_d(\mathcal{B}) := \Big\{ & k = (k_1, k_2, \ldots, k_n) \in \mathbf{Z}^n \;\big|\; \text{for each } i = 1, 2, \ldots, n \text{ there exist} \\ & k' = (k'_1, \ldots, k'_n), k'' = (k''_1, \ldots, k''_n) \in \mathcal{B} \text{ such that } k'_i - d \leq k_i \leq k''_i + d \Big\}. \end{aligned}$$

One easily sees that $|\mathrm{rct}_0(\mathcal{B})|$ is the smallest convex representable (i.e. being a union of a finite number of grid elements) set containing $|\mathcal{B}|$. In the two dimensional case this is the smallest rectangle containing all squares in $\mathcal{B}$. $|\mathrm{rct}_d(\mathcal{B})|$ is the rectangle obtained from $|\mathrm{rct}_0(\mathcal{B})|$ by moving all its edges apart by $d\alpha$.

Let $\delta : Z_{\overset{\circ}{\Sigma}_n} \to \mathbf{Z}^+$ be a function. Our goal is to model the dynamics of the physical system with functions of the form

$$\mathcal{T}^\delta(k) = \mathrm{rct}_{\delta(k)}(\mathcal{T}_0(k)) \cap Z_{\overset{\circ}{\Sigma}_n}.$$

Clearly, each map of this form is consistent with the time series data. However, it is not quite clear what the function δ should be. Ideally, it should be chosen in a way which makes it consistent with time series data obtained by means of *any* possible experiment. Obviously, this is impossible in practice, so all we can do is to choose δ in a way which will enable us to make nontrivial statements about the dynamics of the physical system (subsequent sections will show what it really means). Regardless of what the choice of δ is, we shall need another unverifiable assumption to proceed.

Assumption A3. $\lfloor \mathcal{T}^\delta \rfloor$ *is an envelope of* f*.*

4 Conley Index and Lifting of Dynamics

In the previous section, we outlined the construction of a multivalued map which is to model the dynamics of the physical system. Now we shall discuss how that model can be used to obtain information about dynamics of the original system. Our approach is based on the techniques provided by the Conley index theory. Let us begin with presenting basic definitions in the form which we feel is most suitable for the purpose of time series analysis.

Let F be a multivalued map of a topological space D into itself. We start with the following definition which is fundamental for the Conley index theory.

Definition 4.1 *A pair* $Q = (Q_1, Q_0)$, $Q_0 \subset Q_1 \subset D$, *is an index pair for* F *if and only if the following two conditions hold*

1. $F(Q_0) \cap Q_1 \subset Q_0$,
2. $F(Q_1 \setminus Q_0) \cap Q_1 \setminus Q_0 \subset \mathrm{int}(Q_1 \setminus Q_0)$

Let us stress that the interior in the second inclusion in the above definition is taken relative to D.

Let $B_i, i = 1, 2, \ldots, k$, be closed disjoint sets such that $B_1 \cup B_2 \cup \ldots \cup B_k = \mathrm{cl}(Q_1 \setminus Q_0)$. One can define the maps $r_i : Q_1/Q_0 \to Q_1/Q_0$ by the following formula

$$r_i([x]) = \begin{cases} [x] & \text{if } x \in B_i \\ [Q_0] & \text{otherwise,} \end{cases}$$

where by Q_1/Q_0 we mean the *pointed* topological space resulting from Q_1 when points in Q_0 are identified to a single distinguished point, denoted by $[Q_0]$.

Consider the multivalued map $F_Q : Q_1/Q_0 \overrightarrow{\to} Q_1/Q_0$ defined by

$$F_Q([x]) = \begin{cases} \{[y] \mid y \in F(x)\} & \text{if } F(x) \subset Q_1 \\ \{[y] \mid y \in F(x) \cap Q_1\} \cup \{[Q_0]\} & \text{otherwise.} \end{cases}$$

Note that $F_Q([Q_0]) = \{[Q_0]\}$ by condition 1 of Definition 4.1. Let Γ_{F_Q} be its graph, i.e. the pointed space defined by

$$\Gamma_{F_Q} = \Big(\{(x_1, x_2) \mid x_2 \in F_Q(x_1)\}, ([Q_0], [Q_0])\Big).$$

Now, we can define the multivalued functions

$$\Phi_i^* : H^*(Q_1/Q_0) \rightrightarrows H^*(Q_1/Q_0),\ i = 1, 2, \ldots, k$$

by

$$\Phi_i^*(v) := r_i^*\Big((\pi_1^*)^{-1}(\pi_2^*(v))\Big),$$

where $\pi_j : \Gamma_{F_Q} \to Q_1/Q_0,\ j = 1, 2,$ are the projection maps, defined by $\pi_j(x_1, x_2) = x_j$.

Let us note here that there are a lot of important cases in which Φ_i^* become single-valued endomorphisms. Clearly, this happens if F is a single-valued map (in which case, $\Phi_i^* = r_i^* \circ F_Q^*$). The multivalued Conley index theory of [7] can be looked at as a more general way of imposing additional assumptions on F and Q so as to make Φ_i^* single-valued.

Let us state a simple theorem which will play an important role in our approach to lifting topological invariants of dynamics.

Theorem 4.2 *Assume that $g : A \to A$ and $q : A \to D$ are continuous (single-valued) maps such that the following diagram upper-semicommutes*

$$\begin{array}{ccc} A & \xrightarrow{g} & A \\ \downarrow q & & \downarrow q \\ D & \overset{F}{\rightrightarrows} & D \end{array} \tag{1}$$

(recall that this means that $q(g(x)) \in F(q(x))$ for any $x \in A$). Then:

1. *$q^{-1}(Q) = (q^{-1}(Q_1), q^{-1}(Q_0))$ is an index pair for g.*
2. *Let $\bar{q} : q^{-1}(Q_1)/q^{-1}(Q_0) \to Q_1/Q_0$ be the map induced by q. Then,*

$$\bar{q} \times \bar{q}(\Gamma_{g_{q^{-1}(Q)}}) \subset \Gamma_{F_Q}.$$

Proof. 1. By the assumptions,

$$\begin{aligned} g(q^{-1}(Q_0)) \cap q^{-1}(Q_1) &\subset q^{-1}(F(Q_0)) \cap q^{-1}(Q_1) \\ &\subset q^{-1}(F(Q_0) \cap Q_1) \\ &\subset q^{-1}(Q_0) \end{aligned}$$

and

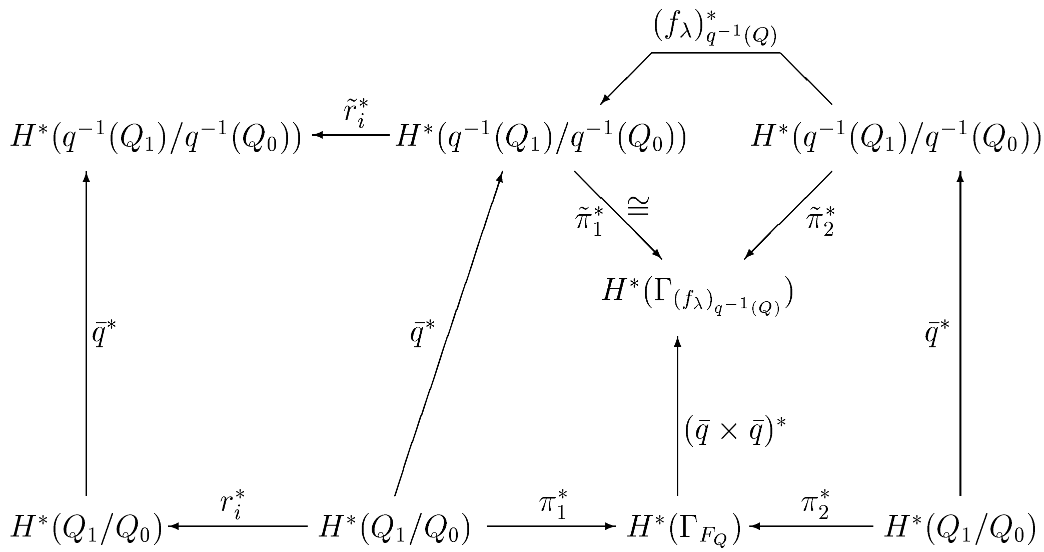
$$\begin{aligned} g(q^{-1}(Q_1)\setminus q^{-1}(Q_0))\cap q^{-1}(Q_1)\setminus q^{-1}(Q_0) &\subset g(q^{-1}(Q_1\setminus Q_0))\cap q^{-1}(Q_1\setminus Q_0) \\ &\subset q^{-1}(F(Q_1\setminus Q_0))\cap q^{-1}(Q_1\setminus Q_0) \\ &\subset q^{-1}(F(Q_1\setminus Q_0)\cap Q_1\setminus Q_0) \\ &\subset q^{-1}(\mathrm{int}(Q_1\setminus Q_0)) \\ &\subset \mathrm{int}(q^{-1}(Q_1)\setminus q^{-1}(Q_0)). \end{aligned}$$

It follows that $q^{-1}(Q)$ is an index pair.
2. The assertion in 2 is equivalent to upper semicommutativity of the diagram

$$\begin{array}{ccc} q^{-1}(Q_1)/q^{-1}(Q_0) & \overset{g_{q^{-1}(Q)}}{\longrightarrow} & q^{-1}(Q_1)/q^{-1}(Q_0) \\ \downarrow \bar{q} & & \downarrow \bar{q} \\ Q_1/Q_0 & \overset{F_Q}{\rightrightarrows} & Q_1/Q_0 \end{array}$$

which follows immediately from upper semicommutativity of (1). □

Let us return to the problem of describing the dynamics of the physical system $f : X \times \Lambda \to X$ now. Let $F = \lfloor T^\delta \rfloor$ be the envelope of f constructed in the preceding section. Recall that F is a multivalued map of $D = |Z_{\overset{\circ}{\Sigma}_n}|$ into itself. By the definition of an envelope, the assumptions of Theorem 4.2 are satisfied for $g = f_{\lambda|A}$ (referred to as f_λ later on), $\lambda \in \Lambda_E$ and $q = \Gamma^{j_0}_{\{\lambda_j\}|A}$. Thus, $q^{-1}(Q)$ is an index pair for f_λ and hence the induced map $(f_\lambda)_{q^{-1}(Q)} : q^{-1}(Q_1)/q^{-1}(Q_0) \to q^{-1}(Q_1)/q^{-1}(Q_0)$ is well-defined. Moreover, if $B_1, B_2, \ldots, B_k$ are closed and disjoint sets whose union is $\mathrm{cl}(Q_1 \setminus Q_0)$ then the sets $\tilde{D}_i = q^{-1}(B_i) \cap \mathrm{cl}(Q_1 \setminus Q_0)$ are closed and disjoint sets whose union is $\mathrm{cl}(q^{-1}(Q_1) \setminus q^{-1}(Q_0))$. Hence, the maps $\tilde{r}_i : q^{-1}(Q_1)/q^{-1}(Q_0) \to q^{-1}(Q_1)/q^{-1}(Q_0)$ which leave the equivalence classes of points in $\tilde{D}_i$ untouched and map all the rest of $q^{-1}(Q_1)/q^{-1}(Q_0)$ into the distinguished point can be defined. We have the following diagram in which $\tilde{\pi}_j$ $(j = 1, 2)$ are the projections from the graph of $(f_\lambda)_{q^{-1}(Q)}$ to $q^{-1}(Q_1)/q^{-1}(Q_0)$.



Let

$$\varphi^*_{\lambda,i} = \tilde{r}_i^* \circ (f_\lambda)^*_{q^{-1}(Q)} : H^*(q^{-1}(Q_1)/q^{-1}(Q_0)) \longrightarrow H^*(q^{-1}(Q_1)/q^{-1}(Q_0)).$$

We have the following proposition.

Proposition 4.3 *For any* $\lambda_0, \lambda_1, \ldots, \lambda_m \in \Lambda_E$ *and* $i_0, i_1, \ldots, i_m \in \{1, 2, \ldots, k\}$,

$$\mathrm{im}\ \bar{q}^* \circ \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m} \subset \mathrm{im}\ \varphi^*_{\lambda_0,i_0} \circ \varphi^*_{\lambda_1,i_1} \circ \ldots \circ \varphi^*_{\lambda_m,i_m} \circ \bar{q}^*.$$

Proof. By the diagram, for any $i \in \{1, 2, \ldots, k\}$ and $\lambda \in \Lambda_E$ and for any subset $V \subset H^*(Q_1/Q_0)$,

$$\begin{aligned}
\bar{q}^* \circ \Phi^*_i(V) &= \bar{q}^* \circ r_i^*((\pi_1^*)^{-1}(\pi_2^*(V))) \\
&= \tilde{r}_i^* \circ \bar{q}^*((\pi_1^*)^{-1}(\pi_2^*(V))) \\
&= \tilde{r}_i^*\Big((\tilde{\pi}_1^*)^{-1}((\bar{q}\times\bar{q})^* \circ \pi_1^*((\pi_1^*)^{-1}(\pi_2^*(V))))\Big) \\
&\subset \tilde{r}_i^*\Big((\tilde{\pi}_1^*)^{-1}((\bar{q}\times\bar{q})^*(\pi_2^*(V)))\Big) \\
&= \varphi^*_{\lambda,i} \circ \bar{q}^*(V).
\end{aligned}$$

An easy inductive argument shows that

$$\bar{q}^* \circ \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m}(V) \subset \varphi^*_{\lambda_0,i_0} \circ \varphi^*_{\lambda_1,i_1} \circ \ldots \circ \varphi^*_{\lambda_m,i_m} \circ \bar{q}^*(V)$$

for any $\lambda_0, \lambda_1, \ldots, \lambda_m \in \Lambda_E$ and $i_0, i_1, \ldots, i_m \in \{1, 2, \ldots, k\}$ and the inclusion in the proposition follows. $\square$

The proposition above motivates the following definition.

$$\Pi^*_{\neq 0} = \Big\{ (i_j)_{j=0}^{\infty} \in \textstyle\prod_{j \in \mathbf{Z}^+} \{1, 2, \ldots, k\} \mid \operatorname{im}\, \bar{q}^* \circ \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m} \neq \{0\} \\ \text{for each } m \in \mathbf{Z}^+ \Big\}$$

(note that 0 is always in the image of $\bar{q}^* \circ \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m}$; in fact, this image is a submodule of $H^*(q^{-1}(Q_1)/q^{-1}(Q_0)))$.

We have the following theorem.

Theorem 4.4 *Any sequence $(i_j)_{j=0}^{\infty} \in \Pi^*_{\neq 0}$ has the following property.*

For any sequence of parameters $\{\mu_j\} \subset \Lambda_E$ and $m \in \mathbf{Z}^+$ there exists an experiment

$$\Big\{ (x_j, \lambda_j) \mid 0 \leq j \leq m \Big\}$$

such that $\lambda_j = \mu_j$, $x_j \in A$ and $\theta(x_j) \cap P_{i_j} \neq \emptyset$ where P_i is the projection of B_i onto the first coordinate, i.e.

$$P_i = \Big\{ x_1 \mid \exists_{x_2, x_3, \ldots, x_n}\ (x_1, x_2, \ldots, x_n) \in B_i \Big\}.$$

Proof. Since

$$\bar{q}^* \circ \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m} \neq \{0\},$$

Proposition 4.3 implies that

$$H^*\Big((f_{\mu_m})_{q^{-1}(Q)} \circ r_{i_m} \circ (f_{\mu_{m-1}})_{q^{-1}(Q)} \circ r_{i_{m-1}} \circ \ldots \circ (f_{\mu_0})_{q^{-1}(Q)} \circ r_{i_0} \Big) = \\ = \varphi^*_{\mu_0, i_0} \circ \varphi^*_{\mu_1, i_1} \circ \ldots \circ \varphi^*_{\mu_m, i_m} \neq 0.$$

Consequently, there is a point $[x] \in q^{-1}(Q_1)/q^{-1}(Q_0)$ such that

$$(f_{\mu_m})_{q^{-1}(Q)} \circ r_{i_m} \circ (f_{\mu_{m-1}})_{q^{-1}(Q)} \circ r_{i_{m-1}} \circ \ldots \circ (f_{\mu_0})_{q^{-1}(Q)} \circ r_{i_0}([x]) \neq [Q_0].$$

Using the argument of [24] (see proof of Theorem 4.4) one shows that the experiment

$$E = \Big\{ (y_j, \mu_j) \mid 0 \leq j \leq m \Big\},$$

where

$$y_j = f_{\mu_{j-1}} \circ f_{\mu_{j-2}} \circ \ldots \circ f_{\mu_0}(x),$$

satisfies $y_j \in \tilde{D}_{i_j}$, so that $q(y_j) \in B_{i_j}$ and therefore $\gamma(y_j) \in P_{i_j}$ (recall that γ is the true measurement which was used in the definition of q). Since $\gamma(y_j) \in \theta(y_j)$, the assertion of the theorem holds for E defined above. □

Alternatively, one can use homology rather than cohomology as a tool to describe the dynamics of f. This leads to the following definitions.

$$\Phi_{i*} : H_*(Q_1/Q_0) \rightrightarrows H_*(Q_1/Q_0),$$

$$\Phi_{i*}(v) = \pi_{2*}\Big((\pi_{1*})^{-1}(r_{i*}(v))\Big),$$

$$\Pi_{\neq 0} = \Big\{(i_j)_{j=0}^{\infty} \in \textstyle\prod_{j\in\mathbf{Z}^+}\{1,2,\ldots,k\} \mid \text{ for each } m \in \mathbf{Z}^+ \text{ there exists } v \in H_*(q^{-1}(Q_1)/q^{-1}(Q_0)) \text{ such that } 0 \notin \Phi_{i_m *} \circ \Phi_{i_{m-1} *} \circ \ldots \circ \Phi_{i_0 *} \circ \bar{q}_*(v)\Big\}.$$

One can easily prove in analogous way as before that any sequence $(i_j)_{j=0}^{\infty} \in \Pi_{\neq 0}$ satisfies the assertion of Theorem 4.4.

Finally, let us mention that, under some admissibility assumptions in the spirit of [20], one can show the existence of an 'infinite experiment' satisfying the hypothesis of the above theorem. To do this, we need the following definitions.

Definition 4.5 The physical system $f : X \times \Lambda \to X$ is called *admissible* iff there exists a sequence $\{\lambda_n\}_{n=-\infty}^{-1} \subset \Lambda_E$ such that for any sequence $\{x_n\}_{n=1}^{\infty}$ of points of A the sequence $(f_{\lambda_{-1}} \circ f_{\lambda_{-2}} \circ \ldots \circ f_{\lambda_{-n}}(x_n))_{n=1}^{\infty}$ has a subsequence convergent to a point in A.

Definition 4.6 A sequence $\{(x_j, \lambda_j) \mid j \in \mathbf{Z}^+\}$ is called an *infinite experiment* if and only if each finite subsequence of its consecutive entries is an experiment.

Define:

$$\Pi_{\neq 0}^{+} = \bigcap_{n=1}^{\infty} \sigma^n(\Pi_{\neq 0}), \qquad \Pi_{\neq 0}^{*+} = \bigcap_{n=1}^{\infty} \sigma^n(\Pi_{\neq 0}^{*}),$$

where σ is the shift map on the set of all one-sided sequences over the alphabet $\{1, 2, \ldots, k\}$. Thus, $\Pi_{\neq 0}^{+}$ and $\Pi_{\neq 0}^{*+}$ are the negatively invariant parts of $\Pi_{\neq 0}$ and $\Pi_{\neq 0}^{*}$ (with respect to the shift map), respectively.

Theorem 4.7 *If f is admissible and $(i_j)_{j=0}^{\infty} \in \Pi^{*+}_{\neq 0}$ then for any sequence $\{\mu_j\}_{j=0}^{\infty} \subset \Lambda_E$ there exists an infinite experiment*

$$\{(x_j, \lambda_j) \mid j \in \mathbf{Z}^+\}$$

such that $\lambda_j = \mu_j$, $x_j \in A$ and $\theta(x_j) \cap P_{i_j} \neq \emptyset$.

Proof. Let $\{\mu_j\}_{j=-\infty}^{\infty}$ be an extension of $\{\mu_j\}_{j=0}^{\infty}$ whose negative part satisfies the conditions of Definition 4.5. For any $n \in \mathbf{N}$ there exists a sequence $\bar{i^n} = (i_j^n)_{j=0}^{\infty} \in \Pi^*_{\neq 0}$ such that, for $\bar{i} = (i_j)_{j=0}^{\infty}$, $\sigma^n(\bar{i^n}) = \bar{i}$. By the same argument as in the proof of Theorem 4.4, there exists $x^n \in A$ such that

$$y_j^n := f_{\mu_{j-n-1}} \circ f_{\mu_{j-n-2}} \circ \ldots \circ f_{\mu_{-n}}(x^n) \in \tilde{D}_{i_j^n}$$

for any $j = 0, 1, \ldots, 2n+1$. By admissibility, the sequence $\{y_n^n\}_{n=1}^{\infty}$ has a cluster point $y_0^* \in A$. It is easy to see that

$$y_j^* := f_{\mu_{j-1}} \circ f_{\mu_{j-2}} \circ \ldots \circ f_{\mu_0}(y_0^*) \in \tilde{D}_{i_j},$$

since y_j^* is a cluster point of the sequence

$$\left\{ y_{j+n}^n = f_{\mu_{j-1}} \circ f_{\mu_{j-2}} \circ \ldots \circ f_{\mu_0}(y_n^n) \right\}_{n=j}^{\infty}$$

all of whose entries are in $\tilde{D}_{i_{j+n}^n} = \tilde{D}_{i_j}$. Clearly, $\{(y_j^*, \mu_j) \mid j \in \mathbf{Z}^+\}$ is an infinite experiment. $\square$

The same proof shows that the hypothesis of the above theorem holds for any sequence in $\Pi^+_{\neq 0}$.

Let us stress that for our approach it is not essential to assume that q is an embedding (this is a very common, often implicitly made assumption in time series analysis). A (rather vague) counterpart of this assumption in our case would be, for example, injectivity of $\bar{q}^*$ (or, dually, surjectivity of $\bar{q}_*$). Notice that, under this assumption, the sets $\Pi_{\neq 0}$ and $\Pi^*_{\neq 0}$ become independent of $\bar{q}$. Namely,

$$\Pi_{\neq 0} = \Big\{ (i_j)_{j=0}^{\infty} \in \textstyle\prod_{j \in \mathbf{Z}^+} \{1, 2, \ldots, k\} \mid \text{ for each } m \in \mathbf{Z}^+ \text{ there exists } w \in H_*(Q_1/Q_0) \text{ such that } 0 \notin \Phi_{i_m *} \circ \Phi_{i_{m-1} *} \circ \ldots \circ \Phi_{i_0 *}(w) \Big\}$$

and

$$\Pi^*_{\neq 0} = \Big\{ (i_j)_{j=0}^{\infty} \in \textstyle\prod_{j \in \mathbf{Z}^+} \{1, 2, \ldots, k\} \mid \text{im } \Phi^*_{i_0} \circ \Phi^*_{i_1} \circ \ldots \circ \Phi^*_{i_m} \neq \{0\} \text{ for each } m \in \mathbf{Z}^+ \Big\}.$$

5 Connectivity and Lifting

In this section we describe a method of detecting chaotic invariant sets that we used for time series data coming from the magnetoelastic ribbon experiment mentioned in the introduction. Although it is not as general as the one discussed in the preceding section, it is much more elementary and usually sufficient in the case when the reconstructed dynamics has only one unstable direction.

Let $F : D \overrightarrow{\to} D$ be the multivalued map taken as a model of the physical system ($D \subset \mathbf{R}^2$) and (Q_1, Q_0) an index pair for F. We shall assume that the sets $Q_0 \subset Q_1 \subset D$ are all representable, i.e. are finite unions of grid elements. Throughout this section, by int, cl and bd we shall mean the interior, closure and boundary relative to D.

Let $\{B_i \mid i = 1, 2, \ldots, k\}$ and $\{C_i \mid i = 0, 1, \ldots, k\}$ be families of pairwise disjoint compact sets whose unions are $\mathrm{cl}(Q_1 \setminus Q_0)$ and $\mathrm{cl}(D \setminus (Q_1 \setminus Q_0))$ (respectively). Define $B_i^+ = B_i \cap C_i$ and $B_i^- = B_i \cap C_{i-1}$ for $i = 1, 2, \ldots, k$. For the rest of this section, we shall assume the following:

(I) $B_i \cap C_j = \emptyset \quad$ if $i \in \{1, 2, \ldots, k\}$ and $j \in \{0, 1, \ldots, k\} \setminus \{i-1, i\}$,

(II) $\mathrm{cl}(Q_1 \setminus Q_0) \cap \mathrm{cl}(D \setminus (Q_1 \setminus Q_0)) \subset Q_0$,

(III) $F(B_i^+) \subset C_{j_i^+}$ and $F(B_i^-) \subset C_{j_i^-}$ for some $j_i^\pm \in \{0, 1, \ldots, k\}$.

Let us note that, by (II), $B_i^\pm \subset Q_0$ and therefore, by the definition of an index pair,

$$F(B_i^\pm) \subset \mathrm{cl}(D \setminus (Q_1 \setminus Q_0)) = \bigcup_{j=0}^{k} C_j.$$

For $i \in \{1, 2, \ldots, k\}$ let

$$\Sigma^i_{\neq 0} = \Big\{ \, (i_l)_{l=0}^\infty \mid i_0 = i \text{ and, for any } l \in \mathbf{Z}^+, \\ \min\{j_{i_l}^+, j_{i_l}^-\} + 1 \le i_{l+1} \le \max\{j_{i_l}^+, j_{i_l}^-\} \, \Big\}$$

and

$$\bar{\Sigma}^i_{\neq 0} = \bigcup_{n \in \mathbf{Z}^+} \sigma^n(\Sigma^i_{\neq 0}).$$

In order to ensure lifting of connectivity information from the finite dimensional model to the phase space X of the physical system we shall need the following set $R \subset \{1, 2, \ldots, k\}$.

$$R = \Big\{ \; i \;\;|\;\; \text{there exists a path } \tau : [0,1] \to A \text{ such that } q(\tau(0)) \in B_i^+ \text{ and } \; q(\tau(1)) \in B_i^- \;\Big\}.$$

Define

$$\bar\Sigma^R_{\neq 0} = \bigcup_{i \in R} \bar\Sigma^i_{\neq 0} \quad \text{and} \quad \bar\Sigma^{R+}_{\neq 0} = \bigcap_{n \in \mathbf{Z}^+} \sigma^n(\bar\Sigma^R_{\neq 0}).$$

Now we are in position to state and prove the main result of this section.

Theorem 5.1 *1. For any sequence of parameters $\{\mu_l\}_{l=0}^\infty \subset \Lambda_E$, $m \in \mathbf{Z}^+$ and $(i_l)_{l=0}^\infty \in \bar\Sigma^R_{\neq 0}$ there exists an experiment*

$$\Big\{ \; (x_j, \lambda_j) \mid 0 \leq j \leq m \; \Big\}$$

such that $\mu_j = \lambda_j$, $x_j \in A$ and $\theta(x_j) \cap P_{i_j} \neq \emptyset$.

2. If f is admissible, $\{\mu_l\}_{l=0}^\infty \subset \Lambda_E$ and $(i_l)_{l=0}^\infty \in \bar\Sigma^{R+}_{\neq 0}$ there exists an infinite experiment

$$\Big\{ \; (x_j, \lambda_j) \mid j \in \mathbf{Z}^+ \; \Big\}$$

such that $\lambda_j = \mu_j$, $x_j \in A$ and $\theta(x_j) \cap P_{i_j} \neq \emptyset$.

Proof. We shall prove that, for any sequence of parameters $\{\mu_j\}_{j=0}^\infty \subset \Lambda_E$, **(*)** for any $j \in R$, any sequence $(i_l) \in \Sigma^j_{\neq 0}$ and $s \in \mathbf{Z}^+$, the composition

$$(f_{\mu_{s-1}})_{q^{-1}(Q)} \circ r_{i_{s-1}} \circ (f_{\mu_{s-2}})_{q^{-1}(Q)} \circ r_{i_{s-2}} \circ \ldots \circ (f_{\mu_0})_{q^{-1}(Q)} \circ r_{i_0}$$

is not the constant map.

An easy consequence of (*) is that the same holds for any sequence in $\bar\Sigma^j_{\neq 0}$ and hence also in $\bar\Sigma^R_{\neq 0}$. Now, by the same argument as in the proofs of Theorems 4.4 and 4.7, both parts of Theorem 5.1 hold. Thus, all we have to do is show (*).

By the definition of R, there exists a path $\tau : [0,1] \to A$ such that $q(\tau(0)) \in B_j^+$ and $q(\tau(1)) \in B_j^-$. Let $\tau_s : [0,1] \to A$ be defined by

$$\tau_s = f_{\mu_{s-1}} \circ f_{\mu_{s-2}} \circ \ldots \circ f_{\mu_0} \circ \tau.$$

We define sequences $\{t_s^1\}_{s=0}^\infty$ and $\{t_s^2\}_{s=0}^\infty$ inductively as follows.

1° Let t_0^1 and t_0^2 be such that $t_0^1 < t_0^2$, $q(\tau_0([t_0^1, t_0^2])) \subset B_{i_0}$, $q(\tau_0(t_0^1)) \in B_{i_0}^+$ and $q(\tau_0(t_0^2)) \in B_{i_0}^-$ (recall that $i_0 = j$ and $\tau_0 = \tau$). Existence of such t_0^1 and t_0^2 follows from (I) and connectivity of the unit interval.

2° Assume that $t_s^1 < t_s^2$ have been defined in such a way that

$$q(\tau_s([t_s^1, t_s^2])) \subset B_{i_s} \tag{2}$$

and

$$\exists_{\alpha\in\{1,2\}} \quad q(\tau_s(t_s^\alpha)) \in B_{i_s}^+ \quad \text{and} \quad q(\tau_s(t_s^{3-\alpha})) \in B_{i_s}^- \tag{3}$$

Since F is an envelope for f, the condition (III) implies

$$q(\tau_{s+1}(t_s^\alpha)) \in C_{j_{i_s}^+} \quad \text{and} \quad q(\tau_{s+1}(t_s^{3-\alpha})) \in C_{j_{i_s}^-}.$$

Again, by condition (I), definition of the $\Sigma_{\neq 0}^j$, compactness of all B_i's and C_i's and connectivity of the unit interval, there exist t_{s+1}^1 and t_{s+1}^2 such that $t_s^1 \le t_{s+1}^1 \le t_{s+1}^2 \le t_s^2$,

$$q(\tau_{s+1}([t_{s+1}^1, t_{s+1}^2])) \subset B_{i_{s+1}}$$

and either

$$q(\tau_{s+1}(t_{s+1}^1)) \in B_{i_{s+1}}^+ \text{ and } q(\tau_{s+1}(t_{s+1}^2)) \in B_{i_{s+1}}^-$$

or

$$q(\tau_{s+1}(t_{s+1}^1)) \in B_{i_{s+1}}^- \text{ and } q(\tau_{s+1}(t_{s+1}^2)) \in B_{i_{s+1}}^+.$$

In this way, we have defined a decreasing sequence $\{[t_s^1, t_s^2]\}$ of subintervals of $[0, 1]$ with properties (2) and (3). Now, let $\bar\tau$ be the loop in Q_1/Q_0 induced by τ, i.e. given by $\bar\tau(t) = [\tau(t)]$ and

$$\bar\tau_s = (f_{\mu_{s-1}})_{q^{-1}(Q)} \circ r_{i_{s-1}} \circ (f_{\mu_{s-2}})_{q^{-1}(Q)} \circ r_{i_{s-2}} \circ \ldots \circ (f_{\mu_0})_{q^{-1}(Q)} \circ r_{i_0} \circ \bar\tau.$$

It is easy to see that, since (2) and (3) hold for all s, $\bar\tau_s(t) = [\tau_s(t)]$ for all $t \in [t_s^1, t_s^2]$. Now, since $B_{i_s}^+$ and $B_{i_s}^-$ are compact and disjoint subsets of B_{i_s}, there is some $t_* \in [t_s^1, t_s^2]$ such that $\tau_s(t_*) \in B_{i_s} \setminus (B_{i_s}^+ \cup B_{i_s}^-)$. It follows that $\bar\tau_s(t_*) \neq [Q_0]$. □

Remark 5.2 Let us note that the proofs of Theorems 4.4, 4.7 and 5.1 show that, in fact, the assertion $\theta(x_j) \cap P_{i_j} \neq \emptyset$ can be replaced with $\gamma(x_i) \in P_{i_j}$ for some true measurement function γ which depends only on f.

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